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Comments on gluon 6-point scattering amplitudes in N=4 SYM at strong coupling

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Abstract: We use the AdS-CFT prescription of Alday and Maldacena [1] to analyze gluon 6-point scattering amplitudes at strong coupling in $\mathcal{N}=4$ SYM. By cutting and gluing we obtain AdS 6-point amplitudes that contain extra boundary conditions and come close to matching the field theory results. We interpret them as parts of the field theory amplitudes, containing only certain diagrams. We also analyze the collinear limits of 6- and 5-point amplitudes and discuss the results.

Keywords: AdS-CFT Correspondence, Gauge-gravity correspondence.

Contents

1. Introduction

The gauge-gravity duality is a valuable tool to investigate the dynamics of gauge theories. Many nonperturbative aspects of gauge theory have been elucidated, mostly for the supersymmetric cases, like $\mathcal{N} = 4$ SYM, where many correlations functions have been analyzed. The thermodynamic and qualitative properties of a large class of gauge theories have been obtained. But until recently, only the properties of gauge invariant states were obtained this way. In [1] however, the amplitudes for scattering of gluons (coloured states) in $\mathcal{N} = 4$ SYM were described using AdS-CFT.

$\mathcal{N} = 4$ SYM and QCD have quite different dynamics at large distances but there are similarities at short distances. The perturbative SYM scattering amplitudes have many features in common with their QCD counterparts, e.g. the SYM loop amplitudes can be considered as components of QCD loop amplitudes (see [2] and references therein). It is thus important to learn as much as possible about the amplitudes of $\mathcal{N} = 4$ SYM, and hope that we can extract information that will be relevant for understanding the QCD physics at hadron colliders.

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Alday and Maldacena [1] proposed a method for computing gluon scattering amplitudes at strong coupling in $\mathcal{N} = 4$ SYM. The essential feature that allowed for the calculation of this coloured amplitude is the factorization of all colour indices into the tree amplitude, $\mathcal{A} = \mathcal{A}_{\text{tree}}\mathcal{M}$, the scalar function $\mathcal{M}$ being calculated from the areas of worldsheets of a classical string in a T dual AdS space. Classical strings are familiar in AdS-CFT from the calculation of Wilson loops. Also, large semiclassical strings correspond to gauge theory operators with large angular momentum [3], or large R charge and spin chain momentum [4], whereas quantum strings correspond to large R charge [5].[1]

Now, the string worldsheet has boundary conditions defined by the gluon states. Gluon states are open strings that end on an infrared D3-brane. The 'T duality on AdS space' was used as a mathematical trick, mapping the open string worldsheet with vertex operators defined by external gauge theory momenta to an open string worldsheet with usual Dirichlet boundary conditions, defined by lightlike segments forming a closed contour, but the AdS space is still noncompact. After the T duality, the boundary and the infrared region are interchanged and so the brane is located on the boundary in the T dual AdS, giving formally the same calculation as for a lightlike Wilson loop. The D3-brane is an infrared regulator in the gauge theory, needed since gluon amplitudes are IR divergent.

Using this prescription, [1] computed the 4-point gluon scattering amplitude at strong coupling in $\mathcal{N} = 4$ at large N, and compared it with the conjectured exact result of Bern, Dixon, and Smirnov (BDS) [8] (see also [9]). The BDS conjecture states that the planar contributions to scattering amplitudes of $\mathcal{N} = 4$ SYM have an iterative structure (at least, for MHV amplitudes): the higher-loop amplitudes are determined by the one-loop amplitude and some functions of the coupling constant.

An n-point amplitude factorizes in two parts: an universal function depending just on momentum invariants times the tree-level amplitude that contains all colors and helicity factors. Unlike in QCD where the scattering amplitudes are very complicated objects, in a SUSY theory the kinematic part is a simple exponential. The four gluon amplitude in $\mathcal{N} = 4$ SYM contains an infrared divergent part plus a finite part that is an elementary function (log squared) of Mandelstam kinematic variables, and is determined by only two functions of the 't Hooft coupling. Thus the only nontrivial information is encoded in these two functions, one of which is related to the cusp anomalous dimension.[2]

The four gluon scattering amplitude computed at strong coupling has the same form as the BDS amplitude at weak coupling with the cusp anomalous value obtained from the semiclassical analysis of [3](see also [11]). Even if the factorization does not hold order by order in the coupling constant for non-MHV amplitudes, it holds again in the strong

[1] The conventional AdS-CFT correspondence relates the strong coupling regime of $\mathcal{N} = 4$ SYM to the supergravity limit of string theory on the $AdS_5 \times S^5$ background for small operators. The analysis of string theory requires large gauge theory operators or, in the spirit of the original 't Hooft string worldsheet proposal, analyzing the zero coupling limit [6, 7].

[2] A nice physical interpretation of the cusp anomaly at weak coupling within the radial quantization approach was given in [10]. That is a *quantum transition amplitude* for a test particle propagating in the radial time and the angular coordinates. Thus, this is an important hint that at strong coupling the correspondent quantity is the *classical action* for a particle propagating on the same phase space.

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coupling limit [1, 12].

One possible reason for the simple form of the conjectured BDS result was explored in [13]: hidden conformal symmetry of the amplitude, not related in an obvious way with the conformal symmetry of the $\mathcal{N}=4$ SYM. Motivated by the work of Alday and Maldacena the authors of [13, 14] investigated the lightlike Wilson loop at weak coupling. They concluded that the duality between gluon amplitudes and Wilson loops is also valid at weak coupling. This is possible evidence for the hidden conformal symmetry of the $\mathcal{N}=4$ SYM. Other recent papers discussing aspects of the Alday-Maldacena proposal are [15 – 18].

In this paper we extend the work of [1] by analyzing 6-point amplitudes. It was explained in [19, 20] that 4- and 5-point amplitudes are fixed by conformal symmetry, and therefore any real test of the BDS conjecture will come for $n=6$ point amplitudes and higher (a conformal Ward identity found in [19] fixes the form of the 4- and 5-point amplitudes, but not higher). In fact, [20] found that a large n calculation gives dissagreement. It is therefore very important to analyze 6-point amplitudes.

We first calculate the strong coupling prediction of the 6-point amplitudes using the BDS conjecture. We then construct 6-point AdS amplitudes by using symmetries and cutting and gluing the 4-point solution. We will see that the lines where we cut and glue actually contain extra boundary conditions, and we will try to interpret them in gauge theory. We will find an interesting relation of these amplitudes to the unitarity cut procedure. The gauge theory 6-point amplitudes we are studying do not have the most general external momenta, and in fact we will obtain a Regge-like behaviour for amplitudes when some of the momentum invariants go to infinity, while others are fixed, similar to the 4-point function behaviour checked by [15].

We will also treat for completeness an 8-point AdS amplitude that can be obtained by the same methods, and interpret it in gauge theory. Finally, we will look at the collinear behaviour of the 6- and 5-point amplitudes to go to the 4-point amplitude. The prescription of [1] implies that it should be possible to get a smooth limit, and we comment how that could be achieved.

The paper is organized as follows: In section 2 we review the calculation of [1]. In section 3 we calculate 6-point amplitudes: first we specify the field theory results, and then we calculate the AdS result and compare. In section 4 we interpret the mismatch and give a gauge theory interpretation of the result. In section 5 we calculate the 8-point amplitude and in section 6 we analyze the collinear limit. An appendix gives some calculational details.

2. Review

Alday and Maldacena [1] describe the 4 dimensional 2 to 2 scattering amplitude for gluons in $\mathcal{N}=4$ SYM. For 2 to 2 scattering of massless particles, there are 4 momenta, each with $E=|\vec{p}|$ ($k^\mu=(E,p^1,p^2,p^3)$). In the center of mass frame, conservation of energy and momentum implies that they are all equal, $E_i=|\vec{p}_i|=k$, $i=1,2,3,4$. As usual, we make all momenta incoming, by changing the sign of the outgoing momenta, so that $\sum_i k_i=0$, and the outgoing momenta have now negative energy. Since the two incoming spatial momenta are parallel, and the two outgoing ones are also parallel (we are in the center of mass frame), we can arange them in a parallelogram, and define k_1, k_2, k_3, k_4 cyclically around the parallelogram. Then the Mandelstam variables are

$$s = -(k_1 + k_2)^2 = -4k^2 \sin^2 \phi/2; \quad t = -(k_1 + k_4)^2 = -4k^2 \cos^2 \phi/2; \quad u = -s - t \tag{2.1}$$

where ϕ is the angle between $\vec{p}_1$ and $\vec{p}_2$, thus s and t are the diagonals of the parallelogram, and $s = t$ corresponds to a square.

In [8], a conjecture was put forth for the gluon scattering amplitudes in $\mathcal{N} = 4$ SYM. We will describe it in more detail in the following section, but for 4-point amplitudes, it is given as follows. The first observation is that the amplitude factorizes as

$$\mathcal{A} = \mathcal{A}_{\rm tree}\mathcal{M}(s,t) \tag{2.2}$$

where $\mathcal{A}_{\rm tree}$ contains all the color and polarization factors, and $\mathcal{M}(s,t)$ is a common function. Then $\mathcal{M}(s,t)$ is written as

$$\begin{aligned} \mathcal{M} &= (\mathcal{A}_{div,s})^2(\mathcal{A}_{div,t})^2 exp\left\{\frac{f(\lambda)}{8}\ln^2\frac{s}{t} + const.\right\} \qquad (2.3)\\ &= exp\left\{-\frac{f(\lambda)}{8}\left(\ln^2\frac{\mu^2}{-s} + \ln^2\frac{\mu^2}{-t}\right) - \frac{g(\lambda)}{2}\left(\ln\frac{\mu^2}{-s} + \ln\frac{\mu^2}{-t}\right) + \frac{f(\lambda)}{8}\ln^2\frac{s}{t} + const.\right\} \end{aligned}$$

where $f(\lambda)$ is the same function appearing in the dimension of twist two operators.

Since the color and polarization factors factorize, we can choose any ordering of k_1, k_2, k_3, k_4 to calculate $\mathcal{M}(s,t)$ (choosing a different ordering will result in a different $\mathcal{A}_{\rm tree}$, but the same $\mathcal{M}(s,t)$). In particular, we will choose the one defined above, with k_1, k_2, k_3, k_4 defined cyclically around the parallelogram of spatial momenta.

The universal function $\mathcal{M}(s,t)$ was obtained in [1] from an AdS space calculation as follows. One starts with AdS_5 space with the metric

$$ds^2 = R^2\frac{d\vec{x}_{3+1}^2 + dz^2}{z^2} \tag{2.4}$$

A Gross-Mende-type calculation [21] for the scattering of open strings dual to the gluons shows that the amplitude is dominated by a classical string worldsheet with vertex operator insertions at the boundary. A 'T duality'

$$\partial_\alpha y^\mu = iw^2(z)\epsilon_{\alpha\beta}\partial_\beta x^\mu \tag{2.5}$$

where neither the initial or the final coordinates are compact gives again AdS space in coordinates

$$ds^2 = R^2\frac{dy_\mu dy^\mu + dr^2}{r^2}; \quad r = \frac{R^2}{z} \tag{2.6}$$

In these T dual coordinates one obtains a classical string worldsheet ending on the boundary at $r = 0$ on a polygon made of lightlike segments dual to the momenta,

$$\Delta y^\mu = 2\pi k^\mu \tag{2.7}$$

Since y_0 is dual to energy, increasing y_0 correponds to incoming momenta and decreasing y_0 to outgoing momenta.

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Then the calculation of $\mathcal{M}(s,t)$ in these T dual variables is formally the same as for the lightlike Wilson loop, i.e.

$$\mathcal{M}(s,t) = e^{iS_{\text{string}}} \sim e^{-\frac{R^2}{2\pi}A} = e^{-\frac{\sqrt{\lambda}}{2\pi}A} \tag{2.8}$$

where A is the area of the minimal string worldsheet, which has Euclidean signature.

In a static gauge $y_1 = u_1, y_2 = u_2$ (where u_1, u_2 are worldsheet coordinates), the string action is

$$iS = -\frac{R^2}{2\pi}\int dy_1 dy_2 \frac{\sqrt{1 + (\partial_i r)^2 - (\partial_i y_0)^2 - (\partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0)^2}}{r^2} \tag{2.9}$$

whereas in a conformal gauge, the action is

$$iS = -\frac{R^2}{2\pi}\int du_1 du_2 \frac{1}{2}\frac{\partial r \partial r + \partial y_\mu \partial y^\mu}{r^2} \tag{2.10}$$

The lightlike contour that the Wilson loop ends on depends on the ordering of external momenta. As we mentioned, we can choose any ordering to calculate $\mathcal{M}(s,t)$, but if we choose the ordering where k_1, k_2 are incoming and k_3, k_4 are outgoing, the projection of the Wilson loop on the y_1, y_2 plane is singular. It is composed of 2 lines, one for the incoming momenta and one for the outgoing ones. That means that choosing $y_1 = u_1, y_2 = u_2$ will be problematic. That is the reason that we choose to define the ordering of k_1, k_2, k_3, k_4 cyclically around the parallelogram of momenta (thus k_1 and k_3 are incoming, and k_2 and k_4 are outgoing).

The worldsheet corresponding to $s = t$ ends on a lightlike polygon, whose projection in the y_1, y_2 plane is a square, thus the boundary conditions are

$$r(\pm 1, y_2) = r(y_1, \pm 1) = 0, \quad y_0(\pm 1, y_2) = \pm y_2; \quad y_0(y_1, \pm 1) = \pm y_1 \tag{2.11}$$

and the solution in static gauge is

$$y_0(y_1, y_2) = y_1 y_2, \quad r(y_1, y_2) = \sqrt{(1 - y_1^2)(1 - y_2^2)} \tag{2.12}$$

or in conformal gauge

$$y_1 = \tanh u_1; \quad y_2 = \tanh u_2; \quad y_0 = \tanh u_1 \tanh u_2; \quad r = \frac{1}{\cosh u_1 \cosh u_2} \tag{2.13}$$

This solution turns out to be the same solution found in [22] for a worldsheet ending on a single lightlike cusp (used for a lightlike Wilson loop calculation).

The solution at $s \neq t$ is obtained by a boost with $b = v\gamma$ in the embedding coordinates of AdS, giving

$$\begin{aligned} y_1 &= \frac{\tanh u_1}{1 + b \tanh u_1 \tanh u_2} & y_2 &= \frac{\tanh u_2}{1 + b \tanh u_1 \tanh u_2} \\ y_0 &= \frac{\sqrt{1+b^2}\tanh u_1 \tanh u_2}{1 + b \tanh u_1 \tanh u_2} & r &= \frac{1}{\cosh u_1 \cosh u_2}\frac{1}{1 + b \tanh u_1 \tanh u_2} \end{aligned} \tag{2.14}$$

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from which one extracts (after a rescaling of momenta by a)

$$s=\frac{-8a^2/(2\pi)^2}{(1-b)^2};\quad t=\frac{-8a^2/(2\pi)^2}{(1+b)^2} \tag{2.15}$$

The two parameters a and b are enough to characterize the amplitude, which is a function of only s and t.

The action on this solution is divergent, indicative of the IR divergence of the gluon amplitude. To deal with it, one introduces a dimensional regularization, $D=4-2\epsilon$, giving the T dual metric

$$ds^2=\sqrt{c_D\lambda_D}\left(\frac{dy_D^2+dr^2}{r^{2+\epsilon}}\right) \tag{2.16}$$

the regularized approximate solution

$$r_\epsilon\sim\sqrt{1+\epsilon/2}\,r_{\epsilon=0};\qquad y^\mu_\epsilon\simeq y^\mu_{\epsilon=0} \tag{2.17}$$

and the action (using that $(\partial r\partial r+\partial y_\mu\partial y^\mu)/(2r^2)|_{\epsilon=0}=1$)

$$S=\frac{\sqrt{\lambda_D c_D}}{2\pi}\int\frac{\mathcal{L}_{\epsilon=0}}{r^\epsilon}=i\frac{\sqrt{\lambda_D c_D}}{2\pi}\int_{-\infty}^{+\infty}du_1du_2r^{-\epsilon}_{\epsilon=0} \tag{2.18}$$
$$\left[1+\frac{\epsilon}{2}\left(\frac{\partial r\partial r}{2r^2}|_{\epsilon=0}-1\right)-\frac{\epsilon^2}{4}\left(\frac{\partial r\partial r}{2r^2}|_{\epsilon=0}-1\right)-\frac{\epsilon^2}{4}\right]$$

The AdS calculation then reproduces the BDS result, giving the values of $f(\lambda)$ and $g(\lambda)$ at strong coupling

$$f=\frac{\sqrt{\lambda}}{\pi};\qquad g=\frac{\sqrt{\lambda}}{2\pi}(1-\ln 2) \tag{2.19}$$

3. Six-point scattering amplitudes

In this section we present six-point scattering amplitudes at strong coupling — these solutions did not appear previously in the literature. We start in the first subsection with a review of BDS conjecture — following [13, 14, 16] we also present a pictorial representation at weak coupling for the finite part of a six-point amplitude. Then, in the next subsection we explictly construct and discuss in detail our new lightlike Wilson loop solutions in AdS.

3.1 Six-point functions: field theory

Bern, Dixon and Smirnov [8] have conjectured more general formulas for the gluon amplitudes, applicable to any n-point function.

The first observation is that the same factorization of color and polarization factors applies for any n-point amplitude, and we have

$$\mathcal{A}_n=\mathcal{A}_n^{\rm tree}\mathcal{M}_n(\epsilon) \tag{3.1}$$

where $\mathcal{M}_n$ only depends on momentum invariants and the dependence on ϵ indicates that we use the dimensional regularization. The supersymmetry constraints the kinematic dependent part to take a nice *exponential* form — specifically, $\mathcal{M}_n(\epsilon)$ can be factorized in an

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infrared divergent part, a finite part, and a coupling-dependent constant:

$$\begin{aligned}
\mathcal{M}_n(\epsilon) &= \mathcal{M}_n^{\rm IR}(\epsilon)F_n(\epsilon)C(\lambda) = \exp\left[\sum_{l=1}^{\infty} a^l f^{(l)}(\epsilon)\hat{I}_n^{(1)}(l\epsilon)\right]\tilde{h}_n(\epsilon) \\
&= \exp\left[\sum_{l=1}^{\infty} a^l f^l(\epsilon)\hat{I}_n^{(1)}(l\epsilon) + \sum_{l=1}^{\infty} a^l f^{(l)}(\epsilon)F_n^{(1)}(l\epsilon) + \sum_{l=1}^{\infty} a^l h_n^{(l)}(\epsilon)\right] \qquad (3.2)
\end{aligned}$$

The constant a is a function of 't Hooft coupling, λ, and the dimensional regularization parameter ϵ:

$$a = \lambda(4\pi e^{-\gamma})^{-\epsilon} \qquad (3.3)$$

where γ is the Euler's constant. In the limit $\epsilon \to 0$, the constant a becomes 't Hooft coupling λ. The functions $f^{(l)}(\epsilon)$ have a perturbative expansion

$$f^{(l)}(\epsilon) = f_0^{(l)} + f_1^{(l)}\epsilon + f_2^{(l)}\epsilon^2 \qquad (3.4)$$

where the first term in expansion, $f_0^{(l)}$, is related to the cusp anomalous dimension for an l-loop. Here $M_n^{(1)}(\epsilon) = I_n^{(1)}(\epsilon) + F_n^{(1)}(\epsilon)$ is the 1-loop amplitude divided by the tree amplitude, thus up to constants and functions of λ the amplitude is the exponential of the 1-loop amplitude. The IR divergent part, $\mathcal{M}_n^{\rm IR}(\epsilon)$, is controlled by the factor $\hat{I}_n^{(1)}(\epsilon)$ that contains $1/\epsilon^2$ IR divergencies. The finite part $F_n(\epsilon)$ that is controlled by the factor $F_n^{(1)}(\epsilon)$ is known as the finite remainder (it is finite as $\epsilon \to 0$), and $h_n^{(l)}(\epsilon)$ are constant factors which do not depend on kinematics.

The divergent factor is

$$\hat{I}_n^{(1)}(\epsilon) = -\frac{1}{2}\frac{1}{\epsilon^2}\sum_{i=1}^{n}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{\epsilon} = -\frac{1}{2\epsilon^2}\sum_{i=1}^{n}\left[1+\epsilon\ln\left(\frac{\mu^2}{-s_{i,i+1}}\right)+\frac{\epsilon^2}{2}\left(\ln\left(\frac{\mu^2}{-s_{i,i+1}}\right)\right)^2+\cdots\right] \qquad (3.5)$$

where $s_{i,i+1} \equiv (k_i + k_{i+1})^2$ are Mandelstam variables for any neighboring pair of gluons and μ is the renormalization scale parameter. Then the amplitude is expanded in ϵ as

$$\begin{aligned}
\ln\mathcal{M}_n(\epsilon) &= \frac{A_2}{\epsilon^2} + \frac{A_1}{\epsilon} + A_0 \\
&-\frac{1}{4}\sum_{i=1}^{n}\sum_{l=1}^{\infty} f_0^{(l)} a^l \left(\ln\left(\frac{\mu^2}{-s_{i,i+1}}\right)\right)^2 - \frac{1}{2}\sum_{i=1}^{n}\sum_{l=1}^{\infty}\frac{f_1^{(l)}}{l}a^l\ln\left(\frac{\mu^2}{-s_{i,i+1}}\right) \\
&+\sum_{l=1}^{\infty} f_0^{(l)} a^l F_n^{(1)}(0) + O(\epsilon) \qquad (3.6)
\end{aligned}$$

where

$$\begin{aligned}
A_2 &= -\frac{n}{2}\sum_{l=1}^{\infty}\frac{f_0^{(l)}}{l^2}a^l \\
A_1 &= -\frac{n}{2}\sum_{l=1}^{\infty}\frac{1}{l^2}f_1^{(l)}a^l - \frac{1}{2}\sum_{l=1}^{\infty}\frac{f_0^{(l)}}{l}a^l\sum_{i=1}^{n}\ln\left(\frac{\mu^2}{-s_{i,i+1}}\right) \\
A_0 &= -\frac{n}{2}\sum_{l=1}^{\infty}\frac{f_2^{(l)}}{l^2}a^l
\end{aligned}$$

Following [17], we define

$$f(\lambda) = 4\sum_{l=1}^{\infty} f_0^{(l)} a^l; \quad g(\lambda) = 2\sum_{l=1}^{\infty} \frac{f_1^{(l)}}{l} a^l \tag{3.7}$$

where $f(\lambda)$ and $g(\lambda)$ are the same functions as defined for the 4-point function. In the limit $\epsilon \to 0$, $f(\lambda) = 4\sum_{l=1}^{\infty} f_0^{(l)} \lambda^l$ is the all-loop cusp anomalous dimension that appears in the dimension of twist two operators. We then obtain for the finite (in ϵ, but still IR divergent in μ) part of the amplitude

$$\ln \mathcal{M}_n|_{\epsilon^0} = A_0 - \frac{1}{16} f(\lambda) \sum_{i=1}^{n} \left(\ln \left(\frac{\mu^2}{-s_{i,i+1}} \right) \right)^2 - \frac{g(\lambda)}{4} \sum_{i=1}^{n} \ln \left(\frac{\mu^2}{-s_{i,i+1}} \right) + \frac{f(\lambda)}{4} F_n^{(1)}(0) \tag{3.8}$$

Finally, the finite remainder for $n > 4$ is given by (for $n = 4$, $F_n^{(1)}(0) = 1/2 \ln^2 s/t$):

$$\frac{f(\lambda)}{4} F_n^{(1)}(0) = \frac{f(\lambda)}{4} \frac{1}{2} \sum_{i=1}^{n} g_{n,i} \tag{3.9}$$

where the functions $g_{n,i}$ contain dilogarithms and squares of ordinary logarithms

$$g_{n,i} = -\sum_{r=2}^{[n/2]-1} \ln \left(\frac{-t_i^{[r]}}{-t_i^{[r+1]}} \right) \ln \left(\frac{-t_{i+1}^{[r]}}{-t_i^{[r+1]}} \right) + D_{n,i} + L_{n,i} + \frac{3}{2}\zeta_2 \tag{3.10}$$

Here we used the 'generalized' Mandelstam variables $t_i^{[r]} \equiv (k_i + \cdots + k_{i+r-1})^2$ (mod n for the index i). The others terms are given by

- $n = 2m + 1$

$$D_{2m+1} = -\sum_{r=2}^{m-1} \mathrm{Li}_2 \left(1 - \frac{t_i^{[r]} t_{i-1}^{[r+2]}}{t_i^{[r+1]} t_{i-1}^{[r+1]}} \right) \tag{3.11}$$

$$L_{2m+1} = -\frac{1}{2} \ln \left(\frac{-t_i^{[m]}}{-t_{i+m+1}^{[m]}} \right) \ln \left(\frac{-t_{i+1}^{[m]}}{-t_{i+m}^{[m]}} \right) \tag{3.12}$$

- $n = 2m$

$$D_{2m} = -\sum_{r=2}^{m-2} \mathrm{Li}_2 \left(1 - \frac{t_i^{[r]} t_{i-1}^{[r+2]}}{t_i^{[r+1]} t_{i-1}^{[r+1]}} \right) - \frac{1}{2} \mathrm{Li}_2 \left(1 - \frac{t_i^{[m-1]} t_{i-1}^{[m+1]}}{t_i^{[m]} t_{i-1}^{[m]}} \right) \tag{3.13}$$

$$L_{2m} = -\frac{1}{4} \ln \left(\frac{-t_i^{[m]}}{-t_{i+m+1}^{[m]}} \right) \ln \left(\frac{-t_{i+1}^{[m]}}{-t_{i+m}^{[m]}} \right) \tag{3.14}$$

and some useful dilogarithmic relations are

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}, \quad \mathrm{Li}_2(0) = 0, \quad \mathrm{Li}_2(1) = \zeta_2 = \frac{\pi^2}{6} \tag{3.15}$$

We can now use the input of the 4-point amplitude AdS calculation of Alday and Maldacena, and substitute the large λ value of $f(\lambda)$ and $g(\lambda)$ in the above formulas. Since

$$\begin{aligned} f(\lambda) &= \frac{\sqrt{\lambda}}{\pi} = 4\sum_{l\geq 1} a^l f_0^{(l)} \\ g(\lambda) &= \frac{\sqrt{\lambda}}{2\pi}(1-\ln 2) = 2\sum_{l\geq 1} \frac{a^l f_1^{(l)}}{l} \end{aligned} \tag{3.16}$$

by acting with $(\lambda d/d\lambda)^{-1}$ once on g and once and twice on f, we get

$$\begin{aligned} &\frac{1}{4}f^{-1}(\lambda) \equiv \sum_{l\geq 1} \frac{a^l f_0^{(l)}}{l} = \frac{\sqrt{\lambda}}{2\pi}; \quad \frac{1}{4}f^{-2}(\lambda) \equiv \sum_{l\geq 1} \frac{a^l f_0^{(l)}}{l^2} = \frac{\sqrt{\lambda}}{\pi} \\ &\frac{1}{2}g^{-1}(\lambda) \equiv \sum_{l\geq 1} \frac{a^l f_1^{(l)}}{l^2} = \frac{\sqrt{\lambda}}{2\pi}(1-\ln 2) \end{aligned} \tag{3.17}$$

Substituting these functions in the amplitude, we get at large coupling (ignoring terms $O(\epsilon)$)

$$\begin{aligned} \ln \mathcal{M}_n = A_0 &- \frac{n\sqrt{\lambda}}{2\pi}\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\left[\frac{n\sqrt{\lambda}}{4\pi}(1-\ln 2) + \frac{\sqrt{\lambda}}{4\pi}\sum_{i=1}^{n} \ln \frac{\mu^2}{-s_{i,i+1}}\right] - \frac{\sqrt{\lambda}}{16\pi}\sum_{i=1}^{n} \ln^2\left(\frac{\mu^2}{-s_{i,i+1}}\right) \\ &- \frac{\sqrt{\lambda}}{8\pi}(1-\ln 2)\sum_{i=1}^{n} \ln\left(\frac{\mu^2}{-s_{i,i+1}}\right) + \frac{\sqrt{\lambda}}{4\pi}F_n^{(1)}(0) \end{aligned} \tag{3.18}$$

For $n = 6$ we have $s_{i,i+1} \equiv t_i^{[2]}$, $t_i^{[3]} = t_{i+3}^{[3]}$ and $t_i^{[4]} = t_{i-2}^{[2]}$ (due to momentum conservation) and then the finite part is given by

$$F_6^{(1)}(0) = -\frac{1}{2}\sum_{i=1}^{6}\left[\ln \frac{t_i^{[2]}}{t_i^{[3]}} \ln \frac{t_{i+1}^{[3]}}{t_i^{[3]}} + \frac{1}{2}\mathrm{Li}_2\left(1 - \frac{t_i^{[2]}t_{i-3}^{[2]}}{t_i^{[3]}t_{i-1}^{[3]}}\right) - \frac{1}{4}\ln^2 \frac{t_i^{[3]}}{t_{i+1}^{[3]}}\right] \tag{3.19}$$

Since $M_6^{(1)}(\epsilon) = I_6^{(1)}(\epsilon) + F_6^{(1)}(\epsilon)$ is the 1-loop amplitude, this formula has an interesting representation. Indeed, the 1-loop amplitude can be written as a sum over box integrals. A nice pictorial representation of this decomposition is to form 'clusters' from external momenta of the 1-loop diagrams and diagonals of the same [16]. The diagonals are then replaced by a partial sum of external momenta and so can be interpreted as off-shell momenta. The clusters with two opposite momenta off-shell and the other two on-shell are called two-mass easy box functions and are usually denoted by $F^{2\mathrm{m\,e}}$ [23]. The clusters with three or four null (on-shell) momenta correspond to one-mass and zero-mass boxes.

For a 6-point amplitude there are two kinds of 4-clusters: the degenerate one ($F_{1;i}^{2me}$) formed from three on-shell external momenta and one off-shell momentum (one diagonal) and the other one ($F^{2me}_{2;i}$) formed from two on-shell external momenta and two off-shell momenta (two diagonals). Thus, we obtain [24, 25] (see also [26]):

$$M_6^{(1)} = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(1-2\epsilon)} \sum_{i=1}^{6}\sum_{r=1}^{2}\Big(1-\frac{1}{2}\delta_{2,r}\Big)\, F^{\mathrm{2m\,e}}_{r;i}(p,q,P,Q) \tag{3.20}$$

where $p = p_{i-1}$, $q = p_{i+r}$, $P = p_i + \cdots + p_{i+r-1}$, and $p+q+P+Q = 0$.

An useful form (all-orders in ϵ) of the two-mass easy box function is given by [26]

$$\begin{aligned} F^{\mathrm{2me}}(s,t,P^2,Q^2) = -\frac{1}{\epsilon^2}\Big[&\Big(\frac{-s}{\mu^2}\Big)^{-\epsilon} + \Big(\frac{-t}{\mu^2}\Big)^{-\epsilon} \\ &-\Big(\frac{-P^2}{\mu^2}\Big)^{-\epsilon} - \Big(\frac{-Q^2}{\mu^2}\Big)^{-\epsilon} + \Big(\frac{a\mu^2}{1-aP^2}\Big)^{\epsilon}{}_2F_1\Big(\epsilon,\epsilon,1+\epsilon,\frac{1}{1-aP^2}\Big) \\ &+\Big(\frac{a\mu^2}{1-aQ^2}\Big)^{\epsilon}{}_2F_1\Big(\epsilon,\epsilon,1+\epsilon,\frac{1}{1-aQ^2}\Big) \\ &-\Big(\frac{a\mu^2}{1-as}\Big)^{\epsilon}{}_2F_1\Big(\epsilon,\epsilon,1+\epsilon,\frac{1}{1-as}\Big)\Big(\frac{a\mu^2}{1-at}\Big)^{\epsilon}{}_2F_1\Big(\epsilon,\epsilon,1+\epsilon,\frac{1}{1-at}\Big)\Big]. \end{aligned} \tag{3.21}$$

where

$$a = \frac{P^2+Q^2-s-t}{P^2Q^2-st} \tag{3.22}$$

and $s := (P+p)^2$, $t := (P+q)^2$.

The first line is the divergent part of the two-mass easy box function that matches the divergent part of the on-shell up to a factor of 2 [13]. After taking the limit $\epsilon \to 0$, the finite part contains only the following dilogarithms [27, 25]

$$\mathrm{Li}_2(1-aP^2) + \mathrm{Li}_2(1-aQ^2) - \mathrm{Li}_2(1-as) - \mathrm{Li}_2(1-at) \tag{3.23}$$

The degenerate cluster (one-mass function) does not contribute to the dilogarithmic part of the BDS formula and since the 4- and 5-point amplitudes only contain this cluster, these amplitudes do not contain dilogarithmic terms.

The duality between lightlike Wilson loops and gluon amplitudes holds also in the weak coupling limit. Thus, to make connection with the Wilson loop computations at strong coupling it would be interesting to understand the MHV amplitudes from a Wilson loop computation at weak coupling. There are two one-loop corrections to the Wilson loop. When the gluon stretches between two lightlike momenta meeting at a cusp there is a contribution to the infrared divergent part of the amplitude. When the gluon stretches between two non-adjacent segments there is a contribution to the finite part.

We will see in the next sections that the AdS-CFT dual amplitudes have extra restrictions, that should correspond to restrictions on the allowed Feynman diagrams in the amplitude. Clearly, these conditions can modify the above cluster decomposition for $\mathcal{M}_n$.

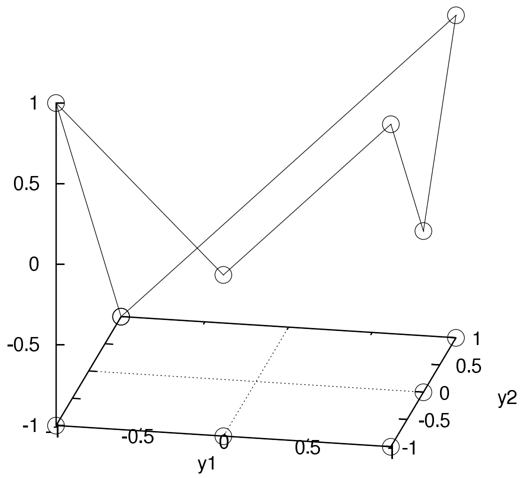

Figure 1: Solution 1: $y_0 = 1/2(|y_1 y_2| + y_1 y_2 - |y_1| y_2 + y_1 |y_2|)$.

3.2 Six-point amplitudes: AdS

The Alday-Maldacena solution for the Wilson loop ending on a square in y_1, y_2 is given in (2.12), and is a solution of the action (2.9) with boundary conditions (2.11). We use the symmetries of the action to construct new simple solutions. Thus, by cutting and gluing these solutions and a careful consideration of the boundary conditions we construct 6-point function solutions of the same action.

First, by noticing that we can change the sign of y_0 in (2.9), we can construct the solution

$$y_0(y_1, y_2) = y_1 |y_2|, \qquad r(y_1, y_2) = \sqrt{(1-y_1^2)(1-y_2^2)} \tag{3.24}$$

(solution 2 in the following) and also a 'composed' solution (solution 1 in the following)

$$\begin{aligned} y_0(y_1, y_2) &= \frac{1}{2}(|y_1 y_2| + y_1 y_2 - |y_1| y_2 + y_1 |y_2|), \\ r(y_1, y_2) &= \sqrt{(1-y_1^2)(1-y_2^2)} \end{aligned} \tag{3.25}$$

The boundary conditions for these solutions are drawn in figure 1 and figure 2, from where we see that they indeed are 6-point functions.

Another 6-point function solution is found by replacing $y_2 \to -2 + |y_2|$ in the Alday-Maldacena solution (and shifting y_0 for convenience), i.e.

$$y_0 - 2 = (-2 + |y_2|) y_1; \qquad r^2 = (1 - y_1^2)(1 - (-2 + |y_2|)^2) \tag{3.26}$$

which again takes advantage of the symmetries of the action and gluing. One can check that the external (incoming and outgoing) momenta are the same for this solution as for the $y_0 = y_1 |y_2|$ solution, just with a different colour ordering. We argued that we can choose any colour ordering to calculate $\mathcal{M}(s,t)$ and we will get the same function. Indeed, since these 2 solutions have the same action (they were obtained by symmetries and cutting and

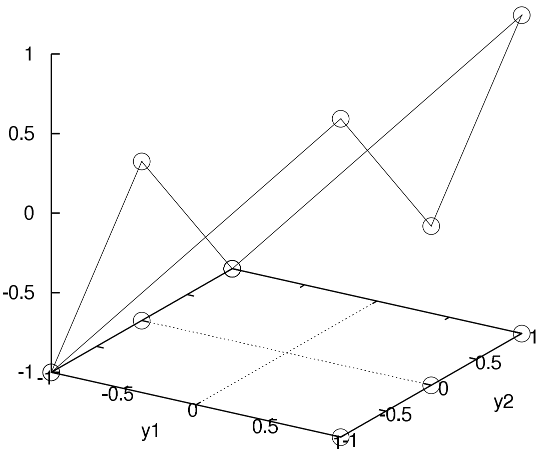

Figure 2: Solution 2: $y_0 = y_1|y_2|$.

gluing), they do give the same result. The external momenta will be in principle different at nonzero b, but we will not analyze this solution further.

Note that the new solutions are not guaranteed to be valid on the lines where we glue them. We will come back to this point at the end of this section, but for the moment we will ignore it.

At this point the new solutions are just a trivial redefining of the old one, but we now need to find the solution for varying external momenta. In the case of the 4-point function, there were only 2 invariant variables, s and t, and consequently we could obtain them from a boost parameter b in the auxiliary embedding coordinate of AdS and an overall scaling by a. For the 6-point function, these two parameters are not enough, since we have more external momenta. In fact there are 8 variables: 6 momenta, minus the center of mass momentum, minus the one momentum given by momentum conservation give 4 momenta. The mass shell conditions of the 4 momenta, spatial rotations, and the mass shell condition of the sum of 5 momenta reduce it to 8 variables.

But what we can do is to make the same transformation as for the 4-point function, depending on parameters a and $b = v\gamma$. We go to the AdS embedding coordinates

$$\begin{aligned} Y^\mu &= \frac{y^\mu}{r} & & (\mu = 0, \dots, 3), \\ Y_{-1} + Y_4 &= \frac{1}{r}, & Y_{-1} - Y_4 &= \frac{r^2 + y_\mu y^\mu}{r}. \end{aligned} \tag{3.27}$$

and perform a Lorentz boost in the 04 plane,

$$\begin{pmatrix} Y'^0 \\ Y'^4 \end{pmatrix} = \begin{pmatrix} \gamma & v\gamma \\ v\gamma & \gamma \end{pmatrix} \begin{pmatrix} Y^0 \\ Y^4 \end{pmatrix}, \tag{3.28}$$

with $\gamma = 1/\sqrt{1 - v^2}$ and a rescaling by a, after which the solution becomes (using that $Y_4 \sim 1 - r^2 - y_\mu y^\mu = 0$)

$$r' = \frac{ar(y_1, y_2)}{1 + by_0(y_1, y_2)}, \qquad y'_0 = \frac{a\sqrt{1 + b^2}\, y_0(y_1, y_2)}{1 + by_0(y_1, y_2)}, \qquad y'_i = \frac{a\sqrt{1 + b^2}\, y_i}{1 + by_0(y_1, y_2)} \tag{3.29}$$

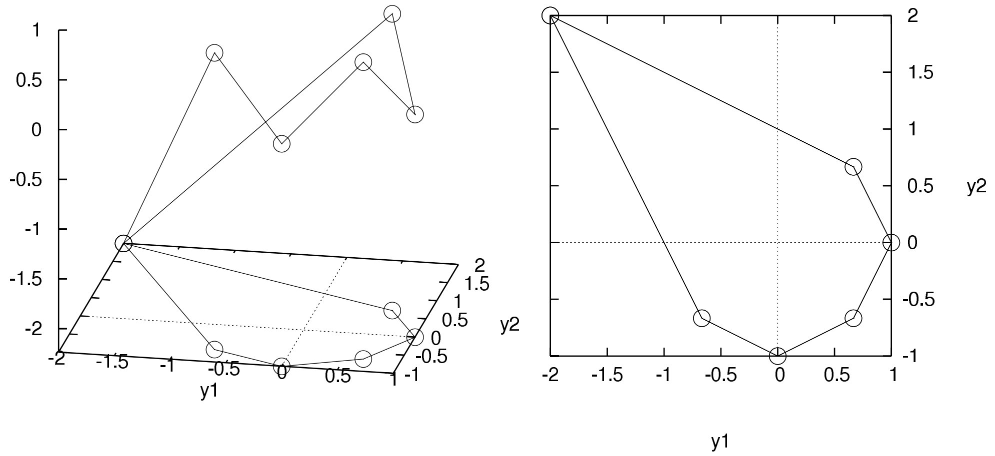

Figure 3: Configuration after the Lorentz boost in the 04 plane for solution 1, $y_0 = 1/2(|y_1 y_2| + y_1 y_2 - |y_1| y_2 + y_1 y_2)$ with $a = 1$, $b = 0.5$.

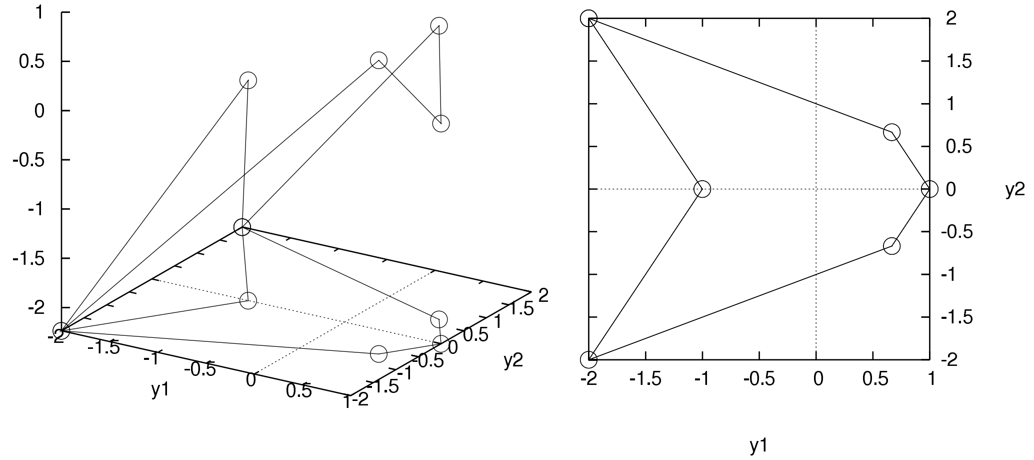

Figure 4: Configuration after the Lorentz boost in the 04 plane for solution 2, $y_0 = y_1 |y_2|$ with $a = 1$, $b = 0.5$.

The boundaries of the boosted solutions (3.25) and (3.24) are depicted in figure 3 and figure 4. In conformal gauge, these solutions are

$$
\begin{aligned}
r &= \frac{a}{\cosh u_1 \cosh u_2 \pm b \sinh u_1 \sinh u_2}, \quad y_0 = \frac{\pm a\sqrt{1+b^2} \sinh u_1 \sinh u_2}{\cosh u_1 \cosh u_2 \pm b \sinh u_1 \sinh u_2} \\
y_1 &= \frac{a \sinh u_1 \cosh u_2}{\cosh u_1 \cosh u_2 \pm b \sinh u_1 \sinh u_2}, \quad y_2 = \frac{a \cosh u_1 \sinh u_2}{\cosh u_1 \cosh u_2 \pm b \sinh u_1 \sinh u_2}
\end{aligned}
\tag{3.30}
$$

where $-/+$ corresponds to $\{u_1 > 0, u_2 < 0\}/(\text{others})$ for the solution $y_0(y_1, y_2) = 1/2(|y_1 y_2| + y_1 y_2 - |y_1| y_2 + y_1 |y_2|)$, and to $\{u_2 > 0\}/\{u_2 < 0\}$ for the solution $y_0(y_1, y_2) = y_1 |y_2|$.

We can read off the external momenta corresponding to these solutions by going to the boundary and defining $k^i = (\Delta y_1^{(i)}, \Delta y_2^{(i)}, \Delta y_0^{(i)})$, where $\Delta y_\mu^{(i)} \equiv y'_\mu(P_{i+1}) - y'_\mu(P_i)$. Here P_i are the vertices of the boundary Wilson line, specifically P_1, P_2, P_3, P_4, P_5 and

P_6 correspond to $(y_1, y_2) = (-1,1)$, $(-1,-1)$, $(0,-1)$, $(1,-1)$, $(1,0)$ and $(1,1)$ in the original coordinate before the boost, for the solution $y_0(y_1,y_2) = 1/2(|y_1 y_2| + y_1 y_2 - |y_1| y_2 + y_1 |y_2|)$, and to $(y_1, y_2) = (-1,1)$, $(-1,0)$, $(-1,-1)$, $(1,-1)$, $(1,0)$ and $(1,1)$ for the solution $y_0(y_1,y_2) = y_1|y_2|$. We then obtain the momenta

$$\begin{aligned}
k_1 &= \left(\frac{2ab}{1-b^2}, -\frac{2a}{1-b^2}, \frac{2a\sqrt{1+b^2}}{1-b^2}\right), \quad & k_2 &= \left(\frac{a}{1+b}, -\frac{ab}{1+b}, -\frac{a\sqrt{1+b^2}}{1+b}\right),\\
k_3 &= \left(\frac{a}{1+b}, \frac{ab}{1+b}, \frac{a\sqrt{1+b^2}}{1+b}\right), & k_4 &= \left(\frac{ab}{1+b}, \frac{a}{1+b}, -\frac{a\sqrt{1+b^2}}{1+b}\right),\\
k_5 &= \left(-\frac{ab}{1+b}, \frac{a}{1+b}, \frac{a\sqrt{1+b^2}}{1+b}\right), & k_6 &= \left(-\frac{2a}{1-b^2}, \frac{2ab}{1-b^2}, -\frac{2a\sqrt{1+b^2}}{1-b^2}\right)
\end{aligned} \tag{3.31}$$

for the solution $y_0(y_1,y_2) = 1/2(|y_1 y_2| + y_1 y_2 - |y_1| y_2 + y_1 |y_2|)$, and

$$\begin{aligned}
k_1 &= \left(\frac{ab}{1-b}, \frac{a}{b-1}, \frac{a\sqrt{b^2+1}}{1-b}\right), & k_2 &= \left(\frac{ab}{b-1}, \frac{a}{b-1}, \frac{a\sqrt{b^2+1}}{b-1}\right)\\
k_3 &= \left(-\frac{2a}{b^2-1}, -\frac{2ab}{b^2-1}, -\frac{2a\sqrt{b^2+1}}{b^2-1}\right), \quad & k_4 &= \left(\frac{ab}{b+1}, \frac{a}{b+1}, -\frac{a\sqrt{b^2+1}}{b+1}\right)\\
k_5 &= \left(-\frac{ab}{b+1}, \frac{a}{b+1}, \frac{a\sqrt{b^2+1}}{b+1}\right) & k_6 &= \left(\frac{2a}{b^2-1}, -\frac{2ab}{b^2-1}, \frac{2a\sqrt{b^2+1}}{b^2-1}\right)
\end{aligned} \tag{3.32}$$

for the solution $y_0(y_1,y_2) = y_1|y_2|$.

We note that the sum of the incoming momenta (if $b < 1$), $k_1 + k_3 + k_5$, is $a/(1-b^2)(1+b^2, -(1+b^2), 2(2-b)\sqrt{1+b^2})$ for solution 1 and $2a/(1-b^2)(1+b^2, 0, 2\sqrt{1+b^2})$ for solution 2, so both are not in the center of mass frame.

We now calculate the AdS amplitude as the exponential of the string action. Since we still have $(\partial r \partial r + \partial y_\mu \partial y^\mu)/(2r^2)|_{\epsilon=0} = 1$ for the new solutions, the dimensionally regularized action on the solution is still (2.19). The dimensionally regularized solution is again (2.17), i.e.

$$r_\epsilon \sim \sqrt{1+\epsilon/2}\, r_{\epsilon=0}; \qquad y^\mu_\epsilon \simeq y^\mu_{\epsilon=0} \tag{3.33}$$

The leading term in (2.19) is then

$$-iS = \frac{\sqrt{\lambda_D c_D}}{2\pi} \int_{-\infty}^{\infty} du_1 du_2 (\cosh u_1 \cosh u_2 + \beta \sinh u_1 \sinh u_2)^\epsilon \tag{3.34}$$

where $\beta = \mp b$ for $\{u_1 > 0, u_2 < 0\}/(\text{others})$ for the solution $y_0(y_1,y_2) = 1/2(|y_1 y_2| + y_1 y_2 - |y_1| y_2 + y_1 |y_2|)$, and $\{u_2 > 0\}/\{u_2 < 0\}$ for the solution $y_0(y_1,y_2) = y_1|y_2|$. We have calculated the subleading terms and they give constant finite contributions as in the [1] case, therefore we will drop them (since we are not considering these constant terms).

The details of the evaluation of the integral are given in the appendix. For the solution $y_0(y_1, y_2) = 1/2(|y_1 y_2| + y_1 y_2 - |y_1|y_2 + y_1|y_2|)$ we obtain

$$\begin{aligned} I &= \int_{-\infty}^{\infty} du_1 du_2 (\cosh u_1 \cosh u_2 + \beta \sinh u_1 \sinh u_2)^{\epsilon} \\ &= \frac{\pi \Gamma[-\frac{\epsilon}{2}]^2}{\Gamma[\frac{1-\epsilon}{2}]^2} \, {}_2F_1\left(\frac{1}{2}, -\frac{\epsilon}{2}, \frac{1-\epsilon}{2}; b^2\right) + \frac{2b}{\epsilon} \, {}_3F_2\left(1, 1, \frac{1-\epsilon}{2}; \frac{3}{2}, 1 - \frac{\epsilon}{2}; b^2\right) \end{aligned} \tag{3.35}$$

where the first term in the last line corresponds to the 4-point function result, and the second is a new contribution. For the solution $y_0(y_1, y_2) = y_1|y_2|)$, we obtain only the first term, thus the same result as for the 4-point function. Using the expansion of the hypergeometric functions,

$$\begin{aligned} {}_2F_1\left(\frac{1}{2}, -\frac{\epsilon}{2}, \frac{1-\epsilon}{2}; b^2\right) &= 1 + \frac{1}{2}\ln(1-b^2)\epsilon + \frac{1}{2}\ln(1-b)\ln(1+b)\epsilon^2 + \mathcal{O}(\epsilon^3) \\ {}_3F_2\left(1, 1, \frac{1-\epsilon}{2}; \frac{3}{2}, 1-\frac{\epsilon}{2}; b^2\right) &= \frac{1}{2b}\ln\left(\frac{1+b}{1-b}\right) + \frac{1}{2b}\left\{-\ln 2 \ln\left(\frac{1+b}{1-b}\right) - \mathrm{Li}_2\left(\frac{1-b}{2}\right)\right. \\ &\qquad \left. + \mathrm{Li}_2\left(\frac{1+b}{2}\right)\right\}\epsilon \end{aligned} \tag{3.36}$$

we obtain the AdS result

$$\begin{aligned} -\frac{\sqrt{\lambda}}{2\pi}\left(2\pi^2 \frac{\mu^2}{4a^2}\right)^{\epsilon/2} &\left[\frac{4}{\epsilon^2} + \frac{2}{\epsilon}\ln(1-b^2) + \frac{2}{\epsilon}(1-\ln 2)\right. \\ &+ (1-\ln 2)\ln(1-b^2) + 2\ln(1-b)\ln(1+b) \\ &\left. + \frac{1}{\epsilon}\ln\frac{1+b}{1-b} + \frac{1+\ln 2}{2}\ln\frac{1+b}{1-b} - \mathrm{Li}_2\left(\frac{1-b}{2}\right) + \mathrm{Li}_2\left(\frac{1+b}{2}\right)\right] \end{aligned} \tag{3.37}$$

for the $y_0(y_1, y_2) = 1/2(|y_1 y_2| + y_1 y_2 - |y_1|y_2 + y_1|y_2|)$ solution, where the first two lines are the 4-point function result and the last line is the extra term. For the $y_0(y_1, y_2) = y_1|y_2|)$ solution, we have only the first two lines, i.e. the 4-point function result. Note that the normalization of μ^2 by $2\pi^2$ is the same as in [1] (part of it is a $(2\pi)^2$ in $t_i^{[2]}$'s in (2.7) and (2.15), and also a factor of 2). The contributions to the action from the higher order ϵ terms in (2.19) is evaluated in a similar way. For the solution 1, +1 is added in the square bracket in (3.37) which is same as the 4-point case. For the solution 2, $1 + b$ is added.

We will now apply our 6-point function field theory formulas for the momenta in (3.31) and (3.32) and compare with the AdS results. For these momenta, the relevant $t_i^{[r]}$ variables are given by

$$\begin{aligned} &t_1^{[2]} = \frac{4a^2}{1-b}, \quad t_2^{[2]} = \frac{4a^2}{(b+1)^2}, \quad t_3^{[2]} = 2a^2, \quad t_4^{[2]} = \frac{4a^2}{(b+1)^2}, \quad t_5^{[2]} = \frac{4a^2}{1-b}, \quad t_6^{[2]} = \frac{8a^2}{(b+1)^2} \\ &t_1^{[3]} = \frac{4a^2}{1-b^2}, \quad t_2^{[3]} = \frac{4a^2}{b+1}, \quad t_3^{[3]} = \frac{4a^2}{b+1}, \quad t_4^{[3]} = \frac{4a^2}{1-b^2}, \quad t_5^{[3]} = \frac{4a^2}{b+1}, \quad t_6^{[3]} = \frac{4a^2}{b+1} \end{aligned} \tag{3.38}$$

for solution 1 ($y_0(y_1, y_2) = 1/2(|y_1 y_2| + y_1 y_2 - |y_1| y_2 + y_1 |y_2|)$) and

$$t_1^{[2]} = \frac{4a^2}{(1-b)^2}, \; t_2^{[2]} = \frac{4a^2}{b+1}, \; t_3^{[2]} = \frac{4a^2}{1-b}, \; t_4^{[2]} = \frac{4a^2}{(b+1)^2}, \; t_5^{[2]} = \frac{4a^2}{1-b}, \; t_6^{[2]} = \frac{4a^2}{b+1},$$
$$t_1^{[3]} = \frac{4a^2}{1-b^2}, \quad t_2^{[3]} = 4a^2, \quad t_3^{[3]} = \frac{4a^2}{1-b^2}, \; t_4^{[3]} = \frac{4a^2}{1-b^2}, \quad t_5^{[3]} = 4a^2, \quad t_6^{[3]} = \frac{4a^2}{1-b^2} \tag{3.39}$$

for solution 2 ($y_0(y_1, y_2) = y_1|y_2|$). Note that for both solutions, if $b < 1$, all the $t_i^{[2,3]}$'s are positive.

We then obtain

$$\begin{aligned} F_6^{(1)}(0) &= \ln 2 \ln(1-b) - 2\ln 2 \ln(1+b) \\ &\quad -2\ln(1-b)\ln(1+b) + \frac{1}{2}(\ln(1-b))^2 + 3(\ln(1+b))^2, \end{aligned} \tag{3.40}$$

for solution 1 and

$$F_6^{(1)}(0) = \frac{3}{2}\left\{(\ln(1-b))^2 + (\ln(1+b))^2\right\} - 2\ln(1-b)\ln(1+b), \tag{3.41}$$

for solution 2.

The divergent piece of the amplitude becomes

$$\begin{aligned} -\frac{\sqrt{\lambda}}{2\pi}\frac{6}{\epsilon^2}\left(\frac{\mu}{2a}\right)^\epsilon & \left(\left(1+\frac{\epsilon}{2}(1-\ln 2)\right)\left(1+\frac{\epsilon}{6}\ln(1-b)(1+b)^3\right)\right. \\ &\left. +\frac{\epsilon^2}{4}\left(\ln^2(1+b) + \frac{1}{6}\ln^2(1-b)\right) + \frac{\epsilon^2}{12}\ln 2\left(\frac{\ln 2}{2} - \ln(1+b)\right)\right) \end{aligned} \tag{3.42}$$

for solution 1 and

$$\begin{aligned} -\frac{\sqrt{\lambda}}{2\pi}\frac{6}{\epsilon^2}\left(\frac{\mu}{2a}\right)^\epsilon & \left(\left(1+\frac{\epsilon}{2}(1-\ln 2)\right)\left(1+\frac{\epsilon}{3}\ln(1-b^2)\right)\right. \\ &\left. +\frac{\epsilon^2}{8}(\ln^2(1+b) + \ln^2(1-b))\right) \end{aligned} \tag{3.43}$$

for solution 2. The finite remainder part can be rewritten as

$$-\frac{\sqrt{\lambda}}{2\pi}\frac{6}{\epsilon^2}\left(\frac{\mu}{2a}\right)^\epsilon\left(-\frac{\epsilon^2}{12}F_6^{(1)}(0)\right) \tag{3.44}$$

Then, reintroducing the general dependence of λ at finite coupling, we can write the total result for these 6-point amplitudes as

$$\begin{aligned} \mathcal{M}_6 = \frac{\mathcal{A}_6}{\mathcal{A}_{6,tree}} &= d(\lambda)\exp\left(-\frac{3}{4\epsilon^2}f^{-2}\left(\lambda\left(\frac{\mu}{2a}\right)^{2\epsilon}\right)\right)\exp\left(-\frac{3}{2}g^{-1}\left(\lambda\left(\frac{\mu}{2a}\right)^{2\epsilon}\right)\right) \\ &\times\left(\frac{1}{b+1}\right)^{\frac{3}{2}g(\lambda)+\frac{3}{4}\frac{f^{-1}(\lambda)}{\epsilon}+\frac{3}{2}f(\lambda)\ln\frac{\mu}{2a}+\frac{f(\lambda)}{4}\ln 2}\left(\frac{1}{1-b}\right)^{\frac{g(\lambda)}{2}+\frac{f^{-1}(\lambda)}{4\epsilon}+\frac{f(\lambda)}{4}\ln\frac{\mu^2}{4a^2}\frac{(1+b)^2}{2}} \end{aligned} \tag{3.45}$$

for solution 1 and

$$\mathcal{M}_6 = \frac{\mathcal{A}_6}{\mathcal{A}_{6,tree}} = d(\lambda) \exp\left(-\frac{3}{4\epsilon^2} f^{-2}\left(\lambda\left(\frac{\mu}{2a}\right)^{2\epsilon}\right)\right) \exp\left(-\frac{3}{2} g^{-1}\left(\lambda\left(\frac{\mu}{2a}\right)^{2\epsilon}\right)\right)$$
$$\times \left(\frac{1}{b+1}\right)^{g(\lambda)+\frac{f^{-1}(\lambda)}{2\epsilon}+f(\lambda)\ln\frac{\mu}{2a}} \left(\frac{1}{1-b}\right)^{g(\lambda)+\frac{f^{-1}(\lambda)}{2\epsilon}+\frac{f(\lambda)}{2}\ln\frac{\mu^2}{4a^2}(1+b)} \tag{3.46}$$

for solution 2. Here $d(\lambda)$ contains finite constant factors. The first line is equal in the two expressions, and is a IR divergent piece depending on the overall scale of the momentum. The second line can be rewritten as

$$\left(\frac{t_2^{[2]}}{2t_3^{[2]}}\right)^{\frac{3}{4}g(\lambda)+\frac{3}{8}\frac{f^{-1}(\lambda)}{\epsilon}+\frac{3}{4}f(\lambda)\ln\frac{\mu}{2a}+\frac{f(\lambda)}{8}\ln 2} \left(\frac{t_1^{[2]}}{2t_3^{[2]}}\right)^{\frac{g(\lambda)}{2}+\frac{f^{-1}(\lambda)}{4\epsilon}+\frac{f(\lambda)}{4}\ln\frac{\mu^2}{2t_2^{[2]}}} \tag{3.47}$$

for solution 1 and

$$\left(\frac{t_2^{[2]}}{t_2^{[3]}}\right)^{g(\lambda)+\frac{f^{-1}(\lambda)}{2\epsilon}+f(\lambda)\ln\frac{\mu}{2a}} \left(\frac{t_3^{[2]}}{t_2^{[3]}}\right)^{g(\lambda)+\frac{f^{-1}(\lambda)}{2\epsilon}+\frac{f(\lambda)}{2}\ln\frac{\mu^2}{t_2^{[2]}}} \tag{3.48}$$

for solution 2. This rewriting is similar to the one performed for the 4-point function in [15], and as there, it relies on the nontrivial cancellation of the leading $\ln^2$ terms between the divergent part and the finite remainder (in this case, $\ln^2(1-b)$ and $\ln^2(1+b)$ terms), without which one could not rewrite the amplitude as this power law.

We can then take the limit $b \to 1$ (a is fixed), which takes several of the $t_i^{[r]}$ parameters to infinity, similar to the s fixed, $t \to -\infty, u \to +\infty$ limit taken by [15], and since the tree amplitude $\mathcal{A}_{6,tree}$ also behaves like a power law, we also get a Regge-like behaviour of the 6-point amplitude $\mathcal{A}_6$, $\sim (t_i^{[2]})^{\alpha(t_2^{[2]})}$, where $t_i^{[2]}$ is a parameter that goes to infinity and $t_2^{[2]}$ stays finite. The physical significance of this result is not clear, but since this power law behaviour doesn't seem to hold for an arbitrary high energy limit (some of the t parameters becoming infinite, others staying finite, and for arbitrary values), it seems to suggest that the cases treated here have a Regge-like explanation as for the 4-point function, in terms of an exchanged particle.

We also observe that if $b < 1$, since all the $t_i^{[2,3]}$'s are positive, the amplitude is real, whereas if $b > 1$ the amplitude becomes complex.

So we have a mismatch between the AdS and field theory results. But all the solutions that we wrote were obtained by cutting and gluing, so there is a potential problem on the line on which we glue. We will try to understand the $y_0 = y_1|y_2|$ solution, since it is easiest, and the mismatch is smallest.

There could be potential delta functions, $\delta(y_2)$, in the equation of motion in static gauge, coming from $\partial_2^2 y_0$. Other than these potential terms, the equations of motion are the same for our solution as for the 4-point function solution, thus are satisfied (since the solutions were obtained by using symmetries of the action).

Since the terms are of the type $\delta(y_2)$, anything multiplied by y_2 gives zero, thus we can put $y_2 = 0$ after taking derivatives. We only look for $\partial_2^2 y_0$ terms, the only ones that give the delta functions. We also substitute $\partial_1 y_0 = 0, \partial_2 r = 0$ after taking derivatives, since both are proportional to y_2.

Then potential delta function terms in the r equation of motion coming from (2.9) are contained in

$$\partial_2 \left[\frac{\partial_2 r + \partial_1 y_0 (\partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0)}{r^2 \sqrt{1 + (\partial_i r)^2 - (\partial_i y_0)^2 - (\partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0)^2}} \right] \tag{3.49}$$

but as we can easily see, after taking derivatives, keeping only $\partial_2^2 y_0$ terms and substituting $y_2 = 0$ as above, we actually get zero. So there are no delta function terms in the r equation of motion.

The y_0 equation of motion is

$$\begin{aligned} &-\frac{1}{r^2}\partial_2 \left[\frac{-2\partial_2 y_0 - 2\partial_1 r(\partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0)}{2r^2 \sqrt{1 + (\partial_i r)^2 - (\partial_i y_0)^2 - (\partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0)^2}} \right] \\ &-\frac{1}{r^2}\partial_1 \left[\frac{-2\partial_1 y_0 + 2\partial_2 r(\partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0)}{2r^2 \sqrt{1 + (\partial_i r)^2 - (\partial_i y_0)^2 - (\partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0)^2}} \right] = 0 \end{aligned} \tag{3.50}$$

and again keeping only $\partial_2^2 y_0$ terms and putting $y_2 = 0$ after taking the derivatives we get, after a bit of algebra, the source (boundary) term

$$\frac{1}{r^2}\partial_2 \left[\frac{\partial_2 y_0 + (\partial_1 r)^2 \partial_2 y_0}{\sqrt{1 + (\partial_i r)^2 - (\partial_i y_0)^2 - (\partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0)^2}} \right] = \frac{y_1 \delta(y_2)}{(1 - y_1^2)^3} \tag{3.51}$$

So one needs to add a source term in the y_0 equation of motion (but not in the r equation of motion) that cancels this term. In other words, we have an extra boundary condition at $y_2 = 0$, a boundary condition in y_0, but not in r. The boundary condition is that $y_0(y_2 = 0) = 0$ (but r is arbitrary).

4. Mismatch interpretation

It is easy to see what would be the interpretation of the boundary condition identified in the previous section. The external boundary of the Wilson loop (for which r=0) is mapped by T duality to physical (on-shell) external momenta of the amplitude. T duality will map the line $y_0 = y_2 = 0$, $\Delta y_1 = 2$ to a momentum k^μ: $(E = 0, p^2 = 0,\ p^1 = 2)$, which is therefore virtual, being spacelike. Moreover, the line $y_2 = 0$ has varying r, which is equal to zero only at the ends. Therefore this momentum is not external (external momenta are defined on the r=0 boundary).

Thus we propose the interpretation that the AdS amplitude we calculated actually corresponds to the following field theory amplitude. Amplitude for three external lines to go into the virtual line k^μ: $(E = 0, p^2 = 0,\ p^1 = 2)$, followed by amplitude for this virtual line to go into other three external lines, as in figure 5a.

It could however also be that there simply is a mismatch between the BDS formula and the dual prescription. Indeed, recently [20] found a mismatch for the $\mathcal{M}_n$ amplitude at large n. They also suggested that since 4- and 5-point amplitudes are determined by

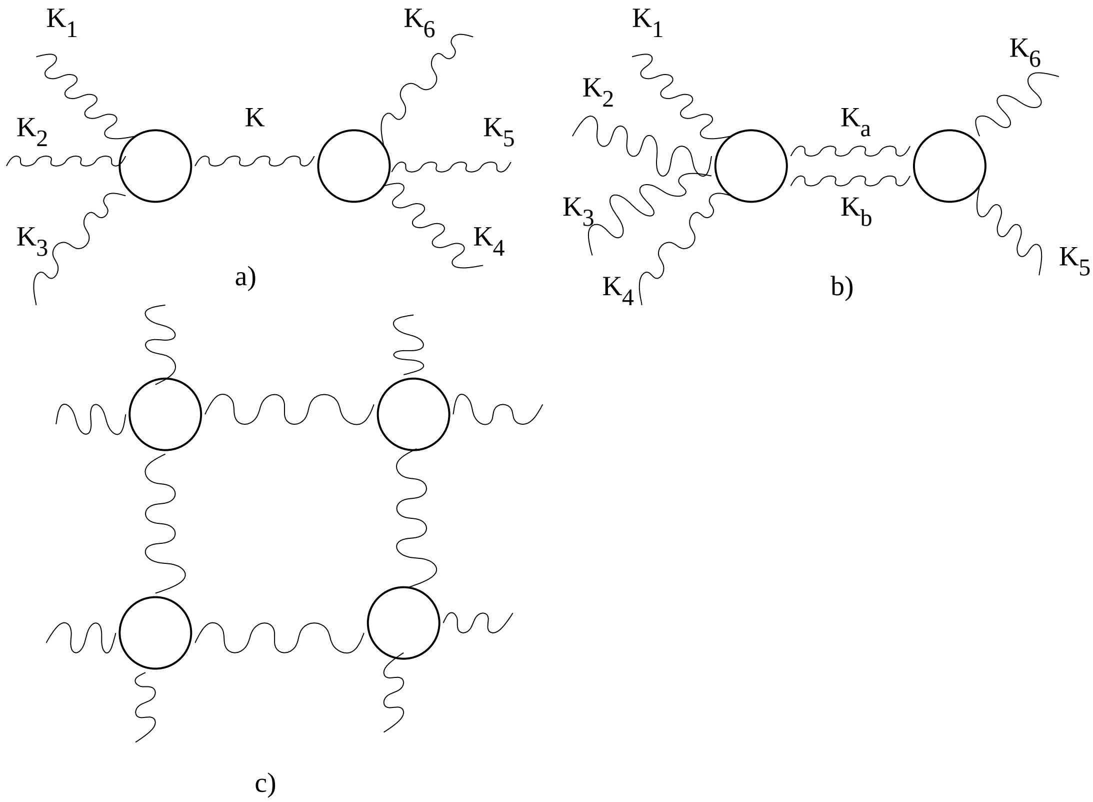

Figure 5: a) Conjectured amplitude calculated by solution 2. b) Conjectured amplitude calculated by solution 1. c) Conjectured amplitude calculated by 8-point function solution.

conformal symmetry [19, 20], there could in principle be dissagreements starting at the 6-point amplitude.

One could ask whether the mismatch between the first line in (3.37) and (3.43) plus (3.44) can be fixed. At first sight this seems encouraging. Indeed, [17] showed that the divergent terms in the BDS formula can be obtained from the contribution near the cusps (corners) of the Wilson loop.

The four corners of the Alday-Maldacena solution have thus the correct behaviour, and they are the same for us, so they are guaranteed to match. But the two extra cusps on the $y_0 = y_2 = 0$ line are potentially problematic. So we could ask whether it is enough to subtract the contribution at our (unsatisfactory) cusps and add the correct cusp behaviour.

The correct cusp behaviour is, according to [17]

$$\sum_{i=1,4}\left(-\frac{\sqrt{\lambda}}{2\pi}\right)\frac{1}{\epsilon^2}C(\epsilon)\left(\frac{\mu^2}{s_{i,i+1}}\right)^{\epsilon/2} \tag{4.1}$$
$$=\left(-\frac{\sqrt{\lambda}}{2\pi}\right)\left(\frac{\mu}{2a}\right)^{\epsilon}\left[\frac{2}{\epsilon^2}+\frac{1-\ln 2}{\epsilon}+\frac{1}{2}\ln^2(1-b)+\frac{1}{2}\ln^2(1+b)\right.$$
$$\left.+\frac{1}{\epsilon}(\ln(1+b)+\ln(1-b))+\frac{1-\ln 2}{2}(\ln(1+b)+\ln(1-b))\right]$$

and we see that at least the b-independent, epsilon-divergent terms are the ones needed for the mismatch.

The contribution of the fake cusps is evaluated in the appendix. It is found to be of order $1/\epsilon$ as needed (since we are missing the $1/\epsilon^2$ term), but the b dependence is not the one we wanted. That means that unfortunately, the missing contribution is not localized at the two fake cusps only.

We then go back to the interpretation of the AdS amplitude as field theory amplitude with an intermediate virtual line and try to understand it better.

If the amplitude we are calculating involves one intermediate virtual line, that means that in order to complete the full 6-point amplitude we are missing amplitudes where the intermediate virtual line is replaced by 2, 3, ... (any number > 1) of intermediate virtual lines.

The separation of the total 6-point amplitude in amplitudes with any number of intermediate lines is familiar from the optical theorem. The optical theorem is a diagramatic equality based on the operatorial relation $-i(T-T^\dagger)=T^\dagger T$, where $T=(S-1)/i$ is the T matrix. The optical theorem states that (twice) the imaginary part of the 6-point amplitude is equal to the sum of the cut amplitudes with 1,2,3,... (any number of) intermediate lines, where cut means putting the lines on-shell, i.e. replacing (for a scalar propagator)

$$\frac{1}{p^2-m^2+i\epsilon}\to -2\pi i\delta(p^2-m^2) \tag{4.2}$$

In order to have such a contribution, we need to have at least an integration over a (loop) momentum for the intermediate lines, which means that the 1 particle cut never contributes.

So the imaginary part of the 6-point amplitude is given by the sum over 2,3,... particle cuts. But the BDS formula states that the 6-point amplitude is real if we have all $t_i^{[2,3]}$'s positive, as is the case for us if $b<1$, therefore the sum of the 2,3,... particle cuts in our case if $b<1$ must be zero.

But the contribution we are missing is one where in the same diagrams we don't cut the propagators, but we compute the whole integral, thus can be potentially nonzero.

Let us also note that we can interpret in a similar manner the 6-point amplitude corresponding to the $y_0(y_1,y_2)=1/2(|y_1y_2|+y_1y_2-|y_1|y_2+y_1|y_2|)$ solution. In a similar way, we see that it has two extra boundaries at $y_1=0=y_0, y_2<0$ and $y_2=0=y_0, y_1>0$, corresponding to 2 spacelike (virtual) momenta $k_a=(0,0,-1)$ and $k_b=(0,1,0)$. Therefore this time the conjectured corresponding field theory amplitude is the amplitude for 4 external lines to go into the two virtual momenta k_a and k_b, followed by the amplitude for the two virtual momenta to go into other two external lines, as in figure 5b).

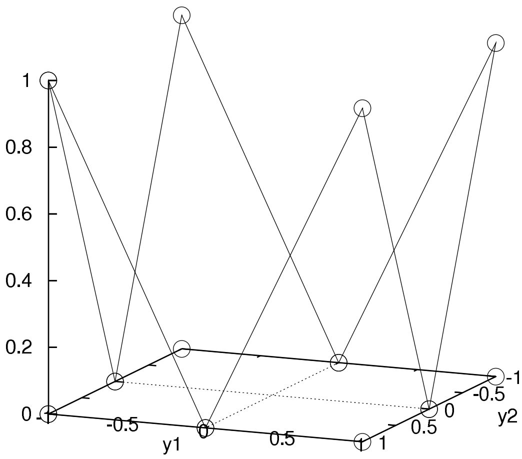

Figure 6: Configuration for 8-point amplitude solution, $y_0 = |y_1 y_2|$

Therefore we conjecture that any extra boundary condition for the Wilson loop, defined on a line, that fixes the y_μ's ($\mu = 0, 1, 2, 3$), but not r, corresponds to an intermediate virtual momentum line with $k^\mu = \Delta y^\mu/(2\pi)$. A priori one can have a Wilson loop with many such boundary conditions, and therefore get an amplitude with many intermediate virtual lines, but then the AdS calculation is probably less useful (it is less useful to know only a very particular set of Feynman diagrams). That is why we have focused on the solution with a single intermediate momentum line.

5. Eight point function

We can generate also an 8-point amplitude from the Alday-Maldacena solution in a manner similar to the 6-point functions. The solution is

$$y_0(y_1, y_2) = |y_1 y_2|; \quad r(y_1, y_2) = \sqrt{(1 - y_1^2)(1 - y_2^2)} \tag{5.1}$$

which is depicted in figure 6. We can again Lorentz boost the solution in the Y_4 embedding coordinate of AdS and obtain the solution drawn in figure 7. It is the same solution as in (3.30), except now the $\pm$ is $= sgn(y_1 y_2)$. From it we can derive the external momenta

$$\begin{aligned}
k_1 &= \left(-\frac{ab}{b+1}, -\frac{a}{b+1}, -\frac{a\sqrt{b^2+1}}{b+1}\right), \qquad & k_2 &= \left(\frac{ab}{b+1}, -\frac{a}{b+1}, \frac{a\sqrt{b^2+1}}{b+1}\right), \\
k_3 &= \left(\frac{a}{b+1}, -\frac{ab}{b+1}, -\frac{a\sqrt{b^2+1}}{b+1}\right), \qquad & k_4 &= \left(\frac{a}{b+1}, \frac{ab}{b+1}, \frac{a\sqrt{b^2+1}}{b+1}\right), \\
k_5 &= \left(\frac{ab}{b+1}, \frac{a}{b+1}, -\frac{a\sqrt{b^2+1}}{b+1}\right), \qquad & k_6 &= \left(-\frac{ab}{b+1}, \frac{a}{b+1}, \frac{a\sqrt{b^2+1}}{b+1}\right), \\
k_7 &= \left(-\frac{a}{b+1}, \frac{ab}{b+1}, -\frac{a\sqrt{b^2+1}}{b+1}\right), \qquad & k_8 &= \left(-\frac{a}{b+1}, -\frac{ab}{b+1}, \frac{a\sqrt{b^2+1}}{b+1}\right).
\end{aligned} \tag{5.2}$$

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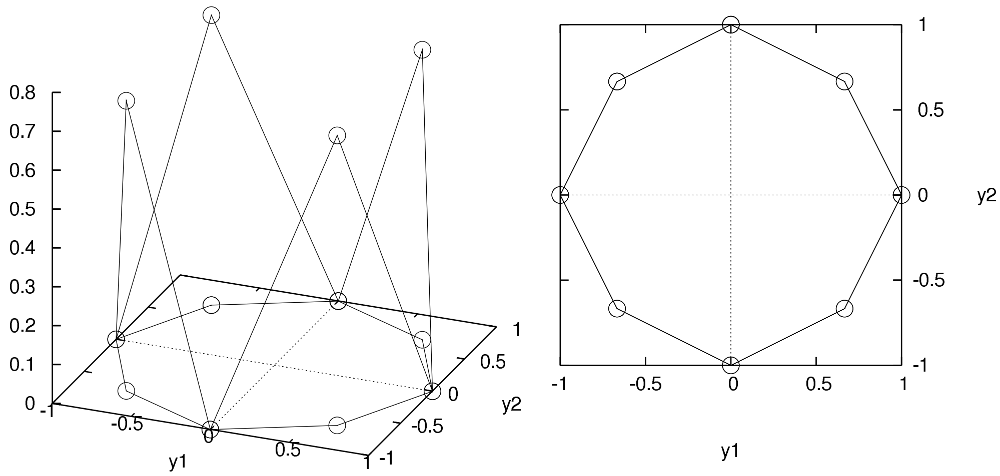

Figure 7: Configuration after the Lorentz boost in the 04 plane for the solution $y_0 = |y_1 y_2|$, with $a = 1$, $b = 0.5$.

The momentum invariants are then

$$t^{[2]}_{\rm odd} = \frac{4a^2}{(b+1)^2}, \quad t^{[2]}_{\rm even} = 2a^2, \quad t^{[3]}_i = \frac{4a^2}{b+1}, \quad t^{[4]}_{\rm odd} = \frac{8a^2}{(b+1)^2}, \quad t^{[4]}_{\rm even} = 4a^2 \tag{5.3}$$

With these values, the finite remainder function is

$$F^{(1)}_8(0) = 4\ln^2(b+1) - 4\ln 2\ln(b+1) - \frac{\pi^2}{6} \tag{5.4}$$

where we have used the relation

$$\mathrm{Li}_2(1/2) = \frac{\pi^2}{12} - \frac{1}{2}(\ln 2)^2. \tag{5.5}$$

The divergent part of the amplitude, from (3.18) is found to be

$$\begin{aligned} \ln \mathcal{M}_{n,div} = & -\frac{4\sqrt{\lambda}}{\pi\epsilon^2}\left(\frac{\mu}{2a}\right)^{\epsilon}\left(\left(1+\frac{\epsilon}{2}(1-\ln 2)\right)\left(1+\frac{\epsilon}{2}\ln(b+1)+\frac{\epsilon\ln 2}{4}\right)\right. \\ & \left. +\frac{\epsilon^2}{16}(4\ln^2(b+1)+\ln^2 2)\right) \end{aligned} \tag{5.6}$$

On the other hand, the AdS result is found to be obtained by multiplying the second term in (3.35) by a factor of 2, thus the final result in (3.37), with the last line multiplied by a factor of 2, i.e.

$$\begin{aligned} -\frac{\sqrt{\lambda}}{2\pi}\left(2\pi^2\frac{\mu^2}{4a^2}\right)^{\epsilon/2} & \left[\frac{4}{\epsilon^2}+\frac{2}{\epsilon}\ln(1-b^2)+\frac{2}{\epsilon}(1-\ln 2)\right. \\ & +(1-\ln 2)\ln(1-b^2)+2\ln(1-b)\ln(1+b) \\ & \left.+\frac{2}{\epsilon}\ln\frac{1+b}{1-b}+2\frac{1+\ln 2}{2}\ln\frac{1+b}{1-b}-2\mathrm{Li}_2\left(\frac{1-b}{2}\right)+2\mathrm{Li}_2\left(\frac{1+b}{2}\right)\right] \end{aligned} \tag{5.7}$$

As evaluated in the appendix, the contribution of the subleading terms give also twice the subleading terms of solution 1 for the 6-point function, thus we have an extra $-2b$ in the square brackets.

The mismatch now is most dramatic, but again the explanation is that we have now 4 extra boundaries on which $y_0 = 0$, namely $y_1 = 0, y_2 > 0$; $y_1 = 0, y_2 < 0$; $y_2 = 0, y_1 > 0$; $y_2 = 0, y_1 < 0$. They will correspond to 4 internal spacelike (virtual) momenta, thus giving the amplitude in figure 5c).

6. Collinear limits

The Alday-Maldacena solution can also be reinterpreted as the collinear limit of a higher n-point amplitude. That is, if we interpret the 4 sides of the Wilson loop not as a single external momentum, but as sets of momenta:

$$k_1 \to \sum_i k_i^{(1)}; \quad k_2 \to \sum_i k_i^{(2)}; \quad k_3 \to \sum_i k_i^{(3)}; \quad k_4 \to \sum_i k_i^{(4)} \tag{6.1}$$

and replace in $s = (k_1 + k_2)^2$ and $t = (k_2 + k_3)^2$. The result for such an n-point amplitude is the Alday-Maldacena result as a function of s and t, now defined as a function of the n momenta $k_i^{(a)}, a = 1, 2, 3, 4$. This is a prediction of the AdS calculation, and we should check that it is indeed obtained from the BDS conjecture. We will see however that there are subtleties related to how we take the limit.

We will now check that the BDS formula for the n-point functions reproduces the 4-point result in the above collinear limit.

Specifically, let us consider the case of 5-point amplitude and take $k_4 = zk_P$ and $k_5 = (1 - z)k_P$, so that $k_P = k_4 + k_5$ and take the limit $k_P^2 \to 0$. This is a usual collinear limit. However, we already see that this is not quite how the limit is taken in string theory. In the AdS computation, we have amplitudes that are already on-shell ($k_4^2 = k_5^2 = 0$), and it is only $2k_4 \cdot k_5 = (k_4 + k_5)^2$ that goes to zero.

For the 5-point function we have

$$g_{5,i} = L_{5,i} = -\frac{1}{2} \ln \left(\frac{t_i^{[2]}}{t_{i+3}^{[2]}} \right) \ln \left(\frac{t_{i+1}^{[2]}}{t_{i+2}^{[2]}} \right) \tag{6.2}$$

and ignoring subleading terms in k_P^2, we have the variables

$$t_1^{[2]} = s_{1,2}; \quad t_2^{[2]} = s_{2,3}; \quad t_3^{[2]} = s_{3,4} = zs_{3,P}; \quad t_4^{[2]} = s_{4,5} = k_P^2; \quad t_5^{[2]} = s_{5,1} = (1 - z)s_{P,1} \tag{6.3}$$

The momenta (k_1, k_2, k_3, k_P) characterize the 4-point amplitude, with variables

$$s_{1,2} = s_{3,P} = -s; \quad s_{1,P} = s_{2,3} = -t \tag{6.4}$$

Then the finite remainder of the 5-point amplitude is

$$\begin{aligned} \frac{f(\lambda)}{4} F_5^{(1)}(0) &= \frac{f(\lambda)}{8} \sum_{i=1}^{5} L_{5,i} \\ &= \frac{f(\lambda)}{8} \left[\ln^2 \frac{s}{t} + \ln \frac{-s}{\mu^2} \ln z + \ln \frac{-t}{\mu^2} \ln(1 - z) - \ln \frac{k_P^2}{\mu^2} \ln z(1 - z) + \ln z \ln(1 - z) \right] \end{aligned} \tag{6.5}$$

where we have introduced an arbitrary scale μ that we want to identify with the IR scale, in order to isolate the finite remainder of the 4-point function, the first term in the last equality.

The divergent piece of the (log of the) 5-point amplitude is

$$
\begin{aligned}
&-\frac{5f^{-2}(\lambda)}{8\epsilon^2}-\frac{5g^{-1}(\lambda)}{4\epsilon}-\left(\frac{f^{-1}(\lambda)}{8\epsilon}+\frac{g(\lambda)}{4}\right)\left[2\ln\frac{\mu^2}{s}+2\ln\frac{\mu^2}{t}-\ln\frac{k_P^2 z(1-z)}{\mu^2}\right]\\
&-\frac{f(\lambda)}{16}\left[2\ln^2\frac{\mu^2}{s}+2\ln^2\frac{\mu^2}{t}+\ln^2\left(\frac{k_P^2 z(1-z)}{\mu^2}\right)-2\ln z\ln(1-z)\right.\\
&\qquad\left.+2\ln\frac{s}{\mu^2}\ln z+2\ln\frac{t}{\mu^2}\ln(1-z)-2\ln\frac{k_P^2}{\mu^2}\ln z(1-z)\right]
\end{aligned}
\tag{6.6}
$$

Adding up the 2 contributions the last line in the divergent piece cancels against the finite remainder and we get the 4-point amplitude with some extra terms

$$
\begin{aligned}
&\ln\mathcal{M}_5\to\ln\mathcal{M}_4-\frac{f^{-2}(\lambda)}{8\epsilon^2}-\frac{g^{-1}(\lambda)}{4\epsilon}+\frac{f(\lambda)}{4}\ln z\ln(1-z)\\
&-\frac{f(\lambda)}{16}\ln^2\frac{k_4\cdot k_5}{\mu^2}+\left(\frac{f^{-1}(\lambda)}{8\epsilon}+\frac{g(\lambda)}{4}\right)\ln\frac{k_4\cdot k_5}{\mu^2}
\end{aligned}
\tag{6.7}
$$

This computation agrees with the one loop result in [24] since at one loop $g(\lambda)=0$ and the extra terms are then

$$
\frac{f(\lambda)}{8}2\ln z\ln(1-z)-\frac{1}{8\epsilon^2}f^{-2}\left(\lambda\left(\frac{\mu^2}{k_4\cdot k_5}\right)^\epsilon\right)
\tag{6.8}
$$

However the extra terms are unfortunate from the point of view of the AdS calculation. The second line in (6.7) dissappears if we take $k_4\cdot k_5=\mu^2$, which is consistent, since both quantities go to zero. The $2\ln z\ln(1-z)$ can be rewritten as $1/2\ln k_4^2/k_P^2\ln k_5^2/k_P^2$ and thus is seen to be due to the fact that k_4^2 and k_5^2 were not zero from the begining, as was the case in the AdS computation. We are still left with the constant terms $-f^{-2}(\lambda)/8\epsilon^2-g^{-1}(\lambda)/4\epsilon$ which arise from the corner of the AdS Wilson loop and thus should dissappear if the collinear limit of the AdS calculation is done correctly (and before taking ϵ to zero).

Next, we consider the 6-point amplitude and take the double collinear limit, $k_1=wk_Q$, $k_2=(1-w)k_Q$, and $k_5=zk_P$ and $k_6=(1-z)k_P$. As before, k_P^2 and k_Q^2 are not zero, but rather go to zero in the collinear limit. In this limit we obtain (dropping subleading k_Q^2 and k_P^2 terms

$$
\begin{aligned}
&t_1^{[2]}=s_{1,2}=k_Q^2; \quad t_2^{[2]}=s_{2,3}=(1-w)s_{Q,3}; \quad t_3^{[2]}=s_{3,4}\\
&t_4^{[2]}=s_{4,5}=zs_{4,P}; \quad t_5^{[2]}=s_{5,6}=k_P^2; \quad t_6^{[2]}=s_{6,1}=(1-z)s_{P,1}=(1-z)ws_{P,Q}\\
&t_1^{[3]}=s_{4,P}; \quad t_2^{[3]}=s_{P,1}=ws_{P,Q}; \quad t_3^{[3]}=(1-z)s_{P,Q}=(1-z)s_{3,4}; \quad t_i^{[3]}=t_{i+3}^{[3]}
\end{aligned}
\tag{6.9}
$$

The momenta (k_P,k_Q,k_3,k_4) characterize the 4-point amplitude, with variables

$$
s_{P,Q}=s_{3,4}=s;\quad s_{Q,3}=s_{P,4}=t
\tag{6.10}
$$

in the limit that k_P and k_Q are on-shell. Then the finite remainder term is

$$\begin{aligned}
\frac{f(\lambda)}{4}F_6^{(1)}(0) &= \frac{f(\lambda)}{8}\sum_{i=1}^{6} g_{6,i} \qquad (6.11)\\
&= \frac{f(\lambda)}{8}\Bigg[\ln^2\frac{s}{t}+\ln\frac{s}{\mu^2}\ln w(1-z)+\ln\frac{t}{\mu^2}\ln z(1-w)-\ln\frac{k_P^2}{\mu^2}\ln z(1-z)\\
&\qquad -\ln\frac{k_Q^2}{\mu^2}\ln w(1-w)+\ln z\ln(1-z)+\ln w\ln(1-w)+\ln w\ln(1-z)\Bigg]
\end{aligned}$$

and the divergent part of the (log of the) 6-point amplitude is

$$\begin{aligned}
&-\frac{3f^{-2}(\lambda)}{4\epsilon^2}-\frac{3g^{-1}(\lambda)}{2\epsilon^2}-\left(\frac{f^{-1}(\lambda)}{8\epsilon}+\frac{g(\lambda)}{4}\right)\left(2\ln\frac{\mu^2}{s}+2\ln\frac{\mu^2}{t}-\ln\frac{k_Q^2}{\mu^2}w(1-w)\frac{k_P^2}{\mu^2}z(1-z)\right)\\
&-\frac{f(\lambda)}{16}\Bigg[2\ln^2\frac{\mu^2}{s}+2\ln^2\frac{\mu^2}{t}+\ln^2\frac{k_P^2}{\mu^2}z(1-z)\\
&\qquad +\ln^2\frac{k_Q^2}{\mu^2}w(1-w)-2\ln z\ln(1-z)-2\ln w\ln(1-w)\\
&\qquad +2\ln w\ln(1-z)+2\ln\frac{t}{\mu^2}\ln z(1-w)+2\ln\frac{s}{\mu^2}\ln w(1-z)\\
&\qquad -2\ln\frac{k_P^2}{\mu^2}\ln z(1-z)-2\ln\frac{k_Q^2}{\mu^2}\ln w(1-w)\Bigg] \qquad (6.12)
\end{aligned}$$

Adding the two contributions the last two lines of the divergent part cancel against terms in the finite remainder and we get

$$\begin{aligned}
&\ln\mathcal{M}_6\to\ln\mathcal{M}_4\\
&+\frac{f(\lambda)}{8}2\ln w\ln(1-w)-\frac{1}{8\epsilon^2}f^{-2}\left(\lambda\left(\frac{\mu^2}{k_1\cdot k_2}\right)^{\epsilon}\right)-\frac{1}{4\epsilon}g^{-1}\left(\lambda\left(\frac{\mu^2}{k_1\cdot k_2}\right)^{\epsilon}\right)\\
&+\frac{f(\lambda)}{8}2\ln z\ln(1-z)-\frac{1}{8\epsilon^2}f^{-2}\left(\lambda\left(\frac{\mu^2}{k_5\cdot k_6}\right)^{\epsilon}\right)-\frac{1}{4\epsilon}g^{-1}\left(\lambda\left(\frac{\mu^2}{k_5\cdot k_6}\right)^{\epsilon}\right) \qquad (6.13)
\end{aligned}$$

i.e., the sum of the contributions of the two collinearities, as expected.

7. Conclusions

In this paper we have analyzed 6-point amplitudes for gluon scattering at strong coupling and large N in $\mathcal{N}=4$ SYM, using AdS-CFT, following the prescription of [1]. We have used the BDS conjecture together with the strong coupling value of the functions $f(\lambda)$ and $g(\lambda)$ calculated in [1] to predict what the AdS results should give. For the AdS calculation, we have analyzed solutions obtained by symmetries, cutting and gluing. We have obtained a mismatch, due to the fact that the AdS solutions contain extra boundary conditions.

The boundary conditions are that $y_0=0$ on an internal line where r is not fixed, and we have interpreted them as having a fixed intermediate virtual momentum line in the amplitude. Thus we propose that the AdS computation calculates only a certain part of the 6-point amplitudes. It would be interesting if one could calculate the gauge theory value for the corresponding amplitude, in order to really test our proposal.

It could also be that there is an actual dissagreement between the BDS conjecture and the dual computation. In [20] it was suggested that a dissagreement could start at n-point amplitudes with $n \geq 6$. The 4- and 5-point amplitudes are fixed by conformal invariance [19, 20], but a dissagreement was found at $n \to \infty$.

The 6-point functions analyzed here do not cover the general external momenta (we have only 2 variables, instead of 8), and in particular we found that for these momenta we obtain a kind of Regge behaviour, where if we take some of the t_i's to infinity by taking $b \to 1$ (which keeps the rest of the t_j's fixed) we have $\mathcal{A} \sim (t_i)^{\alpha(t_j)}$. It would be interesting to understand the physical significance of this result.

We have also treated an 8-point function for completeness, which can be obtained similarly. In this case however, the mismatch is more dramatic, which we understood from our conjectured picture for the extra boundary conditions: the gauge theory amplitude contains only a small part of the possible Feynman diagrams.

The calculation of [1] can be reinterpreted as being a higher n-point amplitude, where the momenta are collinear, such that they form four groups. This implies that there should be a way to take the collinear limit that should avoid extra terms. We have calculated the natural collinear limit of the 5- and 6-point BDS amplitudes, and we have found that we can get rid of most, but not all the extra terms. The issue needs therefore to be understood further, but this can only be done if we have a solution with correct extra cusps (for our solutions, as we saw, the extra cusps did not have the right BDS behaviour).

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A. Integrals

In this appendix we show how to compute integrals necessary for the AdS 6-point amplitudes. The calculation proceeds along the same line as the calculation of the similar integral in the appendix of [1]. First we consider the integral relevant for the leading term (formally of order 1) in (2.19)

$$I = \int_{-\infty}^{\infty} du_1 du_2 (\cosh u_1 \cosh u_2 + \beta \sinh u_1 \sinh u_2)^{\epsilon} \tag{A.1}$$

and expanding in β we get

$$\sum_{l=0}^{\infty}\int_{-\infty}^{+\infty}du_1\int_{-\infty}^{+\infty}du_2\beta^l\frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon+1-l)l!}(\cosh u_1\cosh u_2)^{\epsilon}(\tanh u_1\tanh u_2)^l \tag{A.2}$$

We split the u_1, u_2 integrals into $(-\infty,0)$ and $(0,+\infty)$ and use that $\beta=\pm b$ is constant on those intervals. Then using

$$\int_0^{+\infty}du(\cosh u)^{\epsilon}(\tanh u)^l=\frac{\Gamma\left(\frac{l+1}{2}\right)\Gamma\left(-\frac{\epsilon}{2}\right)}{2\Gamma\left(\frac{1+l-\epsilon}{2}\right)} \tag{A.3}$$

we get for solution 1 ($\pm=-$ if $u_1>0, u_2<0$ and $\pm=+$ otherwise)

$$I=\sum_{l=0}\left(2\left(1+(-1)^l\right)+\left(1-(-1)^l\right)\right)\frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon+1-l)l!}b^l\left(\frac{\Gamma\left(\frac{l+1}{2}\right)\Gamma\left(-\frac{\epsilon}{2}\right)}{2\Gamma\left(\frac{1+l-\epsilon}{2}\right)}\right)^2 \tag{A.4}$$

and doing the sums we get

$$\frac{\pi\Gamma\left[-\frac{\epsilon}{2}\right]^2}{\Gamma\left[\frac{1-\epsilon}{2}\right]^2}\,{}_2F_1\left(\frac{1}{2},-\frac{\epsilon}{2},\frac{1-\epsilon}{2};b^2\right)+\frac{2b}{\epsilon}\,{}_3F_2\left(1,1,\frac{1-\epsilon}{2};\frac{3}{2},1-\frac{\epsilon}{2};b^2\right) \tag{A.5}$$

For this last step write the definitions of the hypergeometric functions as sums and then prove that the terms in the two expressions are the same.

For solution 2, $\pm=+$ if $u_2>0$ and $\pm=-$ if $u_2<0$, we get only the first term in (A.4), i.e. $2(1+(-1)^l)$, and not the $(1-(-1)^l)$ term, and consequently the ${}_3F_2$ term dissappears in the final result.

For the 8-point function solution, $\pm=+$ if $u_1u_2>0$ and $\pm=-$ if $u_1u_2<0$, and we get twice the $(1-(-1)^l)$ term, consequently twice the ${}_3F_2$ term in the final result.

A more general integral, needed for the calculation of the subleading terms is

$$\begin{aligned}I=&\int_0^{\infty}du_1du_2(\cosh u_1\cosh u_2+b\sinh u_1\sinh u_2)^a\times\\&\times\cosh^m u_1\cosh^n u_2\tanh^p u_1\tanh^q u_2=I_{\text{even}}+I_{\text{odd}}\end{aligned} \tag{A.6}$$

Then

$$\begin{aligned}I_{\text{even}}=&\frac{1}{4}B\left(\frac{p+1}{2},-\frac{a+m}{2}\right)B\left(\frac{q+1}{2},-\frac{a+n}{2}\right)\\&\times{}_4F_3\left(\left\{\frac{p+1}{2},\frac{q+1}{2},\frac{1-a}{2}\right\};\left\{\frac{1}{2},\frac{p+1}{2}-\frac{a+m}{2},\frac{q+1}{2}-\frac{a+n}{2}\right\};b^2\right)\\I_{\text{odd}}=&\frac{ab}{4}\frac{\Gamma(-\frac{a+m}{2})\Gamma(-\frac{a+n}{2})}{\Gamma(\frac{2+p-a-m}{2})\Gamma(\frac{2+q-a-n}{2})}\\&\times{}_4F_3\left(\left\{\frac{p+2}{2},\frac{q+2}{2},1-\frac{a}{2},\frac{1-a}{2}\right\};\left\{\frac{3}{2},\frac{p+2-a-m}{2},\frac{q+2-a-n}{2}\right\};b^2\right)\end{aligned} \tag{A.7}$$

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where $a+n, a+m < 0, p+1, q+1 > 0$. For $m = n = p = q = 0$ we get the previous integral, and for $m = 2$, $n = p = q = 0$ we get

$$I = \frac{1}{4\Gamma\left(\frac{1-a}{2}\right)}\left\{\frac{\pi\Gamma\left(-1-\frac{a}{2}\right)\Gamma\left(-\frac{a}{2}\right)}{\Gamma\left(-\frac{1+a}{2}\right)}{}_2F_1\left(\frac{1}{2},-\frac{a}{2},-\frac{1+a}{2};b^2\right)\right.$$
$$\left.+\frac{2^{2+a}(1+a)b\pi\Gamma(-2-a)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(-\frac{a}{2}\right)}{}_3F_2\left(1,1,\frac{1-a}{2};\frac{3}{2},-\frac{a}{2};b^2\right)\right\} \tag{A.8}$$

B. Subleading terms in the action

We write the terms in (2.19) as

$$-iS = B_\epsilon\int_{-\infty}^{+\infty}du_1du_2\frac{1}{(r/a)^\epsilon}(1+\epsilon I_1+\epsilon^2 I_2+\cdots) = B_\epsilon\int_{-\infty}^{+\infty}du_1du_2F_b(u_1,u_2) \tag{B.1}$$

thus the integrand splits as

$$F_b(u_1,u_2) = F_b^{(0)}+\epsilon F_b^{(1)}(u_1,u_2)+\epsilon^2F_b^{(2)}(u_1,u_2)+\cdots \tag{B.2}$$

For the solution 2, we can reduce the integration to integration from 0 to infinity by using the symmetries. We get

$$-iS = B_\epsilon\int_0^\infty\int_0^\infty du_1du_2\left\{F_b(u_1,u_2)+F_b(-u_1,u_2)+F_{-b}(u_1,-u_2)+F_{-b}(-u_1,-u_2)\right\} \tag{B.3}$$

but because $F_b(-u_1,u_2) = F_b(u_1,-u_2) = F_{-b}(u_1,u_2)$ we get

$$-iS = B_\epsilon\int_0^\infty\int_0^\infty du_1du_2\;\; 2\left\{F_b(u_1,u_2)+F_{-b}(u_1,u_2)\right\} \tag{B.4}$$

which is the same result as for the 4-point function. Thus, as is the case there, the subleading terms just give a +1 added inside the square brackets in (3.37).

For the solution 1, we have

$$-iS = B_\epsilon\int_0^\infty\int_0^\infty du_1du_2\left\{F_b(u_1,u_2)+F_b(-u_1,u_2)+F_{-b}(u_1,-u_2)+F_b(-u_1,-u_2)\right\} \tag{B.5}$$

and using the symmetries, we get

$$\begin{aligned}-iS &= B_\epsilon\int_0^\infty\int_0^\infty du_1du_2\left\{2F_b(u_1,u_2)+2F_{-b}(u_1,u_2)+(F_b(u_1,u_2)-F_{-b}(u_1,u_2))\right\}\\ &= -iS^{4-point}+B_\epsilon\int_0^\infty\int_0^\infty du_1du_2(F_b(u_1,u_2)-F_{-b}(u_1,u_2))\end{aligned} \tag{B.6}$$

Then the order ϵ term (from $F_b^{(1)}(u_1,u_2)$) in the difference gives

$$\begin{aligned}-i\Delta S^{(1)} &= (b^2-1)\frac{2^\epsilon(\epsilon-1)b\pi\Gamma(-\epsilon)}{4\Gamma\left(\frac{3-\epsilon}{2}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(-\frac{\epsilon-2}{2}\right)}{}_3F_2\left(1,1,\frac{3-\epsilon}{2};\frac{3}{2}-\frac{\epsilon-2}{2};b^2\right)\\ &\quad -b^2\frac{b}{\epsilon-2}{}_3F_2\left(1,1,\frac{3-\epsilon}{2};\frac{3}{2},2-\frac{\epsilon}{2};b^2\right)\end{aligned} \tag{B.7}$$

which in the limit of $\epsilon \to 0$ becomes $-b/\epsilon + \cdots$. Then from the relation (2.19) we can check that the order ϵ^2 term in the action does not contribute (goes to zero).

Thus for solution 1, the contribution of subleading terms adds up to a $(-b)$ inside the square brackets in (3.37).

For the 8-point function, we get

$$\begin{aligned} -iS &= B_\epsilon \int_0^\infty \int_0^\infty du_1 du_2 \left\{F_b(u_1,u_2) + F_{-b}(-u_1,u_2) + F_{-b}(u_1,-u_2) + F_b(-u_1,-u_2)\right\} \\ &= -iS^{4-pt} + B_\epsilon \int_0^\infty \int_0^\infty du_1 du_2 \;\; 2(F_b(u_1,u_2) - F_{-b}(u_1,u_2)) \end{aligned} \tag{B.8}$$

thus the contribution of the subleading terms is twice that of solution 1.

C. Fake cusp calculation

In this appendix we evaluate the contribution of the fake cusps to the 6-point AdS amplitude defined by $y_0 = y_1|y_2|$.

But in order to do so we must select a method that will reproduce the correct behaviour for a correct cusp.

According to eq. (3.21) in [17] (see also eq. (3.29) in [1]), the contribution to the string action from near a correct cusp is

$$-iS_{i,i+1}(\epsilon) = \frac{\sqrt{\lambda_D c_D}}{8\pi} \frac{\sqrt{1+\epsilon}}{(1+\epsilon/2)^{1+\epsilon/2} 2^{\epsilon/2}} (-s_{i,i+1})^{-\epsilon/2} \int_0^1 \frac{dY_- dY_+}{(Y_- Y_+)^{1+\epsilon/2}} \tag{C.1}$$

where $Y_\pm$ are coordinates parallel to the 2 momenta (sides of the cusps). The integration in the original variables $y_\pm$ was from 0 to the values of the momentum, i.e. the length of the side of the cusp, except that the solution used was not the exact one for the polygon Wilson loop, but rather the approximate one for the infinite cusp. Note that the integral gives $(2/\epsilon)^2$ (it's the product of two identical integrals).

The integration above was done in $y_\pm = y_0 \pm y_1$ variables (with y_2 added), since the solution used was an infinite cusp with lightlike boundary. But the fake cusp we are interested in has not only the lightlike boundary along $\tilde{y}_\pm = y_0 \pm y_2$, but also the boundary $y_0 = 0, y_2 = 0$, so clearly $\tilde{y}_\pm$ are not good integration variables for the cusp solution. Rather, we will use y_1, y_2.

In order to understand the y_1, y_2 integration procedure better, we will first analyze the b=0 solution, looking at both the usual (Alday-Maldacena) cusp and the new fake cusp. The solution is

$$r^2 = (1+\epsilon/2)(1-y_1^2)(1-y_2^2); \quad y_0 = y_1|y_2| \tag{C.2}$$

The square root in the action (2.9) is (after a bit of algebra)

$$\sqrt{1 + \frac{\epsilon}{2}(y_1^2 + y_2^2)} \tag{C.3}$$

Near a good cusp, e.g. $y_1 = y_2 = 1$, we have

$$\begin{aligned} r &\simeq 2\sqrt{\delta y_1 \delta y_2}\sqrt{1+\epsilon/2}; \quad y_0 \simeq 1 - \delta y_1 - \delta y_2 \\ \mathcal{L} &= \frac{\sqrt{1+\epsilon} d\delta y_1 d\delta y_2}{(1+\epsilon/2)^{1+\epsilon/2}(4\delta y_1 \delta y_2)^{1+\epsilon/2}} \end{aligned} \tag{C.4}$$

Then the action at the cusp is

$$-iS_{i,i+1}(\epsilon) = \frac{\sqrt{\lambda_D c_D}}{8\pi} \frac{\sqrt{1+\epsilon}}{(1+\epsilon/2)^{1+\epsilon/2} 2^{\epsilon}} \int_0^1 \frac{d\delta y_1 d\delta y_2}{(\delta y_1 \delta y_2)^{1+\epsilon/2}} \tag{C.5}$$

Near a fake cusp, e.g. $y_2 = 0, y_1 = 1$, we have

$$\begin{aligned} r &\simeq \sqrt{2\delta y_1}\sqrt{1+\epsilon/2}; \quad y_0 \simeq |\delta y_2| \\ \mathcal{L} &= \frac{\sqrt{1+\epsilon/2} d\delta y_1 d\delta y_2}{(1+\epsilon/2)^{1+\epsilon/2}(2\delta y_1)^{1+\epsilon/2}} \end{aligned} \tag{C.6}$$

and the action at the cusp is

$$-iS_{i,i+1}(\epsilon) = \frac{\sqrt{\lambda_D c_D}}{4\pi} \frac{\sqrt{1+\epsilon/2}}{(1+\epsilon/2)^{1+\epsilon/2} 2^{\epsilon/2}} \int_0^1 \frac{d\delta y_1}{(\delta y_1)^{1+\epsilon/2}} \int_{-1}^1 d\delta y_2 \tag{C.7}$$

which now contains a single divergent integral, so is of order $1/\epsilon$, not $1/\epsilon^2$.

Now we turn to the nonzero b case. For nonzero b, the equation of the Alday-Maldacena (4-point function) curve in y_0, y_1, y_2, r coordinates is obtained from (2.14) by writing $\tanh u_1$, $\tanh u_2$ as a function of y_1, y_2 and substituting in r, y_0 with the result

$$\begin{aligned} y_0 &= \frac{\sqrt{1+b^2}}{2b}\left(1 - \sqrt{1 - 4by_1y_2}\right) \\ r &= \frac{1}{2b}\frac{\sqrt{\left[4b^2y_2^2 - (1-\sqrt{1-4by_1y_2})^2\right]\left[4b^2y_1^2 - (1-\sqrt{1-4by_1y_2})^2\right]}}{(1-\sqrt{1-4by_1y_2})} \end{aligned} \tag{C.8}$$

For the 6-point function solution we replace everywhere y_2 by $|y_2|$. Near the fake corner $u_2 = 0, u_1 = +\infty \leftrightarrow y_2 = 0, y_1 = 1$ we get (after some algebra)

$$\begin{aligned} y_0 &\simeq \sqrt{1+b^2}|\delta y_2|(1 - \delta y_1 + b|\delta y_2|) \\ r &\simeq \sqrt{1+\epsilon/2}\sqrt{2(\delta y_1 - b|\delta y_2|)}\left(1 - \frac{\delta y_1 - b|\delta y_2|}{4}\right) \end{aligned} \tag{C.9}$$

and, again after some algebra, we get the Lagrangian

$$\mathcal{L} = \frac{\sqrt{1+\epsilon/2(1+b^2)}}{(1+\epsilon/2)^{1+\epsilon/2}2^{1+\epsilon/2}} \frac{d\delta y_1 d\delta y_2}{(d\delta y_1 - b|\delta y_2|)^{1+\epsilon/2}} \tag{C.10}$$

Now we need to change to variables that are parallel to k_4, k_5. Since the momenta are

$$k_4^\mu = \frac{1}{b+1}(b, 1, \dots); \quad k_5^\mu = \frac{1}{b+1}(b, -1, \dots) \tag{C.11}$$

we get that the new variables Y_1, Y_2 that are parallel to k_4, k_5 and run from 0 to 1 are defined as

$$\begin{aligned} \delta y_1 &= \frac{b}{b+1}(Y_1+Y_2); & \delta y_2 &= \frac{1}{b+1}(Y_1-Y_2) \Rightarrow \\ d\delta y_1 d\delta y_2 &= \frac{2b}{(b+1)^2}dY_1 dY_2; & (\delta y_1 - b|\delta y_2|) &= \frac{2b}{b+1}min\{Y_1, Y_2\} \end{aligned} \tag{C.12}$$

Then the action at the fake cusp is

$$\begin{aligned} -iS_{i,i+1} &= \frac{\sqrt{\lambda_D c_D}}{2\pi}\frac{\sqrt{1+\epsilon/2(1+b^2)}}{(1+\epsilon/2)^{1+\epsilon/2}2^{1+\epsilon/2}}\int\frac{d\delta y_1 d\delta y_2}{(\delta y_1 - b|\delta y_2|)^{1+\epsilon/2}} \\ &= \frac{\sqrt{\lambda_D c_D}}{2\pi}\frac{\sqrt{1+\epsilon/2(1+b^2)}}{(1+\epsilon/2)^{1+\epsilon/2}2^{1+\epsilon/2}}\frac{1}{b+1}\left(\frac{2b}{b+1}\right)^{-\epsilon/2}\frac{-4}{\epsilon(1-\epsilon/2)} \end{aligned} \tag{C.13}$$

If $\epsilon \ln b < 1$ we obtain

$$-iS_{i,i+1} \simeq \frac{\sqrt{\lambda}}{2\pi}\frac{2}{\epsilon}\frac{1}{b+1}\left(\frac{\pi\mu}{a}\right)^{\epsilon}\left(1+\frac{\epsilon}{4}(1+b^2)+\frac{\epsilon}{2}\left(1-\ln\frac{b}{b+1}\right)\right) \tag{C.14}$$

This contribution is indeed of order $1/\epsilon$ as we wanted (since we are missing the $1/\epsilon^2$ term), but the b dependence is incorrect.

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